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EXTENDED CONFORMAL SUPERGRAVITY

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We present the complete structure of extended conformal supergravity for $N \leq 4$. The relation with the graded algebra $SU(2, 2|N)$ and with the multiplet of currents is discussed. The $N = 4$ superconformal theory has a formulation with local $SU(4) \times U(1)$ and rigid $SU(1, 1)$ invariance. We give the linearized invariant actions for all N . For $N = 2$ we exhibit the complete non-linear lagrangian.

1. Introduction

Conformal supergravity has a higher degree of symmetry than any other field theory that is presently known. Its classical action is invariant under conformal transformations, as well as under two different kinds of supersymmetry, called Q and S supersymmetry. For extended supergravity we have N independent invariances of both types. Furthermore the geometry of these theories implies the presence of chiral $U(N)$ as an extra invariance. Nevertheless, not much is known about the structure of the extended theories. For $N = 1$ and 2 the field representations and transformation rules have been given [1–3], but the complete invariant lagrangian has only been constructed for $N = 1$ [1]. Beyond $N = 2$ no field representations are known.

In this paper we study extended conformal supergravity with $N \leq 4$. In view of the difficulties for higher N , which we discuss below, this implies that we give a complete treatment of all superconformal theories of the conventional type. An early account of our work has already been given elsewhere [4]. Our study is not primarily motivated by the hope that conformal theories are directly relevant to the description of elementary particles and their interactions, although this remains an interesting avenue for future research. At present the importance of superconformal gravity stems from the fact that it can be used to gain a deeper and more systematic insight into the structure of Poincaré and de Sitter supergravity theories. This has been fully demonstrated for $N = 2$ supergravity, where superconformal methods were used to construct and explore a variety of supergravity theories [4, 5].

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It seems doubtful that superconformal gravity will exist for $N > 4^*$. Already in [6] it was observed that the lagrangian of conformal supergravity will contain the standard kinetic terms for the $SU(N)$ and $U(1)$ gauge fields, and that beyond $N = 4$ those terms will occur with opposite signs [for $N = 4$ the $U(1)$ field is absent]. These signs are intrinsically related to the graded $SU(2, 2|N)$ algebras. Another indication that drastic changes take place beyond $N = 4$ is provided by the off-shell counting arguments presented in [7]. In that work it was argued that Poincaré supergravity theories should contain auxiliary fields that are able to generate states with spin $s > 2$. These fields occur in the superconformal lagrangian as dynamical fields, so that the same difficulties will arise as in the Poincaré theory with $N > 8^{**}$. To evade the higher-spin fields one could assume the presence of off-shell central charges in the supersymmetry algebra; such a formulation has been proposed in [9]. However, central charges are incompatible with the $U(N)$ symmetry of the conformal theories, so this option seems excluded.

The first step in the construction of superconformal gravity is to find the underlying field representation together with the linearized transformation rules. For $N = 1$ the theory contains only the gauge fields of $SU(2, 2|1)$ subject to certain constraints [1]. But for higher N conformal supergravity will contain matter fields, so that the relation with the graded $SU(2, 2|N)$ algebras can no longer be exploited to derive the full conformal theory. Instead one must follow a different approach. In this paper we have chosen to deduce linearized conformal supergravity from the multiplet of currents [10, 11], which we shall present here for $N = 4$. This supermultiplet deserves some interest in its own right, since it forms an important ingredient in recent attempts to construct a phenomenologically viable unification scenario based on supergravity [12]. Conformal supergravity is still related to the graded $SU(2, 2|N)$ algebras, and the precise relationship is pointed out in the text. Furthermore, we shall show in detail how conformal supergravity can also be represented in terms of the Weyl multiplet [13]. This multiplet is a chiral superfield subject to certain supersymmetric restrictions. Part of these restrictions coincide with the Bianchi identities for the $SU(2, 2|N)$ gauge-field curvatures. Knowledge of the Weyl multiplet allows the immediate construction of the linearized superconformal lagrangian. The relation between the superconformal fields and those of the extended Poincaré theories can then be deduced by making use of off-shell counting arguments [4]. The Weyl multiplet is also expected to represent the on-shell structure of extended Poincaré supergravity (see for instance ref. [14]). In that context it plays a role in the discussion of on-shell invariant counterterms [13, 15]. In connection with the Weyl multiplet we also pay attention to the preservation of

* The upper limit of $N = 4$ for extended conformal supergravity was first conjectured by Gell-Mann (unpublished)

** The difficulty of finding interacting field theories for higher spins has been emphasized in [8], see also references quoted therein

chirality for superfields. The latter is no longer obvious for higher- N supergravity theories.

The linearized results for conformal supergravity can be extended to the full theory by means of iteration. However, it is advantageous to do this in a formulation which exhibits the highest possible degree of invariance. Therefore we first construct a new version of $N = 4$ conformal supergravity which is manifestly symmetric under an extra local $U(1)$ and rigid $SU(1, 1)$ group. In this formulation the complete non-linear results are obtained after a finite number of iterations. In this way we find the superconformal transformations and the corresponding algebra which are presented in the text. In principle, these results allow the construction of the full superconformal lagrangian for $N \leq 4$. We have not attempted to do this here. Instead we present some explicit results for $N = 2$ conformal supergravity. We discuss the $N = 2$ Weyl multiplet including all non-linear modifications. The invariant action is then obtained by making full use of the existing multiplet calculus for $N = 2$ supergravity [3, 4, 16, 17].

This paper is organized as follows. We present the linearized results in sects. 2 and 3. Sect. 2 discusses the relation of conformal supergravity with the $SU(2, 2|N)$ algebra and the multiplet of currents. Sect. 3 introduces the Weyl multiplets, and the superconformal lagrangians. The full non-linear transformations are given in sect. 4. The Weyl multiplet and the lagrangian for $N = 2$ are constructed to all orders in sect. 5. Sect. 6 gives our conclusions. There are two appendices. Appendix A contains a discussion of the $N = 4$ multiplet of currents. Appendix B gives a derivation of some invariance properties of massive supermultiplets.

2. Linearized conformal supergravity

Conformal and superconformal gravity can be formulated as the gauge theory of the (super)conformal algebra $SU(2, 2|N)$ [6, 18]. Here N specifies the number of

TABLE 1
Generators and corresponding gauge fields of the superconformal group

Superconformal gauge symmetries			Gauge fields	
4	translations	P	e_μ^a	45
6	Lorentz rotations	M	ω_μ^{ab}	
1	dilatation	D	b_μ	
4	conformal boosts	K	f_μ^a	24N
4N	supersymmetries	Q	ψ_μ^i	
4N	special supersymmetries	S	ϕ_μ^i	
$N^2 - 1$	chiral $SU(N)$	C	V_μ^i	$3(N^2 - 1)$
1	chiral $U(1)$	A	A_μ	3

The numbers in the right-hand column denote the degrees of freedom represented by the gauge fields. The $U(1)$ symmetry is absent in the case of $N = 4$

independent Q -supersymmetry generators of this graded algebra. Choosing $N = 0$ corresponds to conformal gravity (the Weyl theory), and $1 \leq N \leq 4$ to (extended) conformal supergravity. The algebra $SU(2, 2|N)$ contains the generators of the conformal group $SU(2, 2)$, of both ordinary and special supersymmetry, and of chiral $U(N)$ or $SU(N)$. (For $N = 4$ the graded algebra does not imply the $U(1)$ subgroup of $U(4)$.) We have presented these symmetry generators with their corresponding gauge fields in table 1.

The transformation rules for the superconformal gauge fields follow straightforwardly from the algebra. We give the transformations under Q and S supersymmetry, dilatations (D), special conformal transformations (K), and chiral $U(1)$ transformations, with corresponding parameters ε^i , η^i , Λ_D , Λ_K^a and Λ_A :

$$\begin{aligned}
\delta e_\mu^a &= -\Lambda_D e_\mu^a + (\bar{\varepsilon}^i \gamma^a \psi_{\mu i} + \text{h.c.}) , \\
\delta \omega_\mu^{ab} &= \Lambda_K^{[a} e_\mu^{b]} - (\bar{\varepsilon}^i \sigma^{ab} \phi_{\mu i} + \text{h.c.}) + (\bar{\psi}_\mu^i \sigma^{ab} \eta_i + \text{h.c.}) , \\
\delta b_\mu &= \partial_\mu \Lambda_D + \Lambda_K^a e_\mu^a + \frac{1}{2}(\bar{\varepsilon}^i \phi_{\mu i} + \text{h.c.}) - \frac{1}{2}(\bar{\psi}_\mu^i \eta_i + \text{h.c.}) , \\
\delta f_\mu^a &= \mathcal{D}_\mu \Lambda_K^a + \Lambda_D f_\mu^a + \frac{1}{2}(\bar{\eta}^i \gamma^a \phi_{\mu i} + \text{h.c.}) , \\
\delta \psi_\mu^i &= 2\mathcal{D}_\mu \varepsilon^i - \frac{1}{2}\Lambda_D \psi_\mu^i - \gamma_\mu \eta^i - i \frac{4-N}{4N} \Lambda_A \psi_\mu^i , \\
\delta \phi_\mu^i &= 2\mathcal{D}_\mu \eta^i + \frac{1}{2}\Lambda_D \phi_\mu^i - 2f_\mu^a \gamma_a \varepsilon^i + \Lambda_K^a \gamma_a \psi_\mu^i + i \frac{4-N}{4N} \Lambda_A \phi_\mu^i , \\
\delta V_{\mu j} &= \bar{\varepsilon}^i \phi_{\mu j} - \bar{\psi}_\mu^i \eta_j - \frac{1}{N} \delta^i_j (\bar{\varepsilon}^k \phi_{\mu k} - \bar{\psi}_\mu^k \eta_k) - \text{h.c.} , \\
\delta A_\mu &= i(\bar{\varepsilon}^i \phi_{\mu i} - \text{h.c.}) - i(\bar{\psi}_\mu^i \eta_i - \text{h.c.}) + \partial_\mu \Lambda_A .
\end{aligned} \tag{2.1}$$

We have used a chiral $SU(N)$ notation [17], where ψ_μ^i , ϕ_μ^i , ε^i and η_i have positive chirality and transform according to the defining N -dimensional representation of $SU(N)$; $\bar{\psi}_\mu^i$, $\bar{\phi}_\mu^i$, ε_i and η^i have negative chirality and transform according to the conjugate representation. The derivative \mathcal{D}_μ is covariant with respect to Lorentz and chiral $(S)U(N)$ transformations and dilatations. The curvatures corresponding to the gauge fields have been listed in table 2.

The gauge fields of $SU(2, 2|N)$ also transform under general coordinate transformations, which are not related to the $SU(2, 2|N)$ transformations. This is because the $SU(2, 2|N)$ symmetries have been treated so far as an “internal” symmetry group, i.e., as symmetries that do not act on space and time (although their parameter values are not necessarily the same at each space–time point). This is the reason why the commutator of two supersymmetry transformations does not yield a general coordinate transformation, but a P transformation. Hence we do not have a supersymmetry algebra of the conventional type, and for this reason the numbers of

TABLE 2
Curvatures of the graded $SU(2, 2|N)$ algebra

$R_{\mu\nu}{}^a(P) = \mathcal{D}_{[\mu} e_{\nu]}{}^a - \frac{1}{2} \bar{\psi}_{[\mu}{}^i \gamma^a \psi_{\nu]i},$
$R_{\mu\nu}{}^{ab}(M) = \partial_{[\mu} \omega_{\nu]}{}^{ab} - \omega_{[\mu}{}^{ac} \omega_{\nu]}{}^{cb} - f_{[\mu}{}^{[a} e_{\nu]}{}^{b]} + \frac{1}{2} (\bar{\psi}_{[\mu}{}^i \sigma^{ab} \phi_{\nu]i} + \text{h.c.}),$
$R_{\mu\nu}(D) = \partial_{[\mu} b_{\nu]} - f_{[\mu}{}^a e_{\nu]}{}^a - \frac{1}{4} (\bar{\psi}_{[\mu}{}^i \phi_{\nu]i} + \text{h.c.}),$
$R_{\mu\nu}{}^a(K) = \mathcal{D}_{[\mu} f_{\nu]}{}^a - \frac{1}{4} \bar{\phi}_{[\mu}{}^i \gamma^a \phi_{\nu]i},$
$R_{\mu\nu}{}^i(Q) = \mathcal{D}_{[\mu} \psi_{\nu]}{}^i - \frac{1}{2} \gamma_{[\mu}{}^a \phi_{\nu]}{}^i,$
$R_{\mu\nu}{}^i(S) = \mathcal{D}_{[\mu} \phi_{\nu]}{}^i + f_{[\mu}{}^a \gamma_a \psi_{\nu]}{}^i,$
$R_{\mu\nu}(A) = \partial_{[\mu} A_{\nu]} - \frac{1}{2} i (\bar{\psi}_{[\mu}{}^i \phi_{\nu]i} - \text{h.c.}),$
$R_{\mu\nu}{}^i{}_j(V) = \partial_{[\mu} V_{\nu]}{}^i{}_j - V_{[\mu}{}^i{}_k V_{\nu]}{}^k{}_j - \frac{1}{2} (\bar{\psi}_{[\mu}{}^i \phi_{\nu]i} - (1/N) \delta^i{}_j \bar{\psi}_{[\mu}{}^k \phi_{\nu]k} - \text{h.c.}).$

bosonic and fermionic degrees of freedom of the $SU(2, 2|N)$ gauge fields are not equal (see table 1).

To convert the P transformations into general coordinate transformations requires imposing constraints. For instance, it is easy to show that a P transformation on the (vierbein) field $e_{\mu}{}^a$ coincides with a general coordinate transformation combined with some of the remaining symmetries of $SU(2, 2|N)$ if the curvature $R(P)$ is set to zero. However, on imposing this constraint the gauge field $\omega_{\mu}{}^{ab}$ ceases to be an independent field, and is fully determined in terms of other gauge fields. In principle one could now investigate whether it is possible to require that P transformations can be expressed in terms of the other transformations in a uniform manner on all gauge fields. This will lead to further constraints, which may again determine some of the gauge fields. For instance, for $N=0$ one can show that P transformations become dependent on other symmetries, provided one imposes one more constraint to determine the conformal gauge field $f_{\mu}{}^a$. The most obvious choice for this constraint is

$$R_{\mu\nu}{}^{ab}(M) e^{\nu}{}_b = 0. \quad (2.2)$$

However, we emphasize that this constraint is rather arbitrary; any constraint that determines $f_{\mu}{}^a$ is sufficient.

This argument may be continued to higher N , but the constraints will become too strong to allow for supergravity! To resolve this one recognizes that a uniform conversion of P transformations into general coordinate transformations is no longer the crucial point. The presence of the constraints has caused essential changes in the underlying geometry of the theory. This is so because the constraints determine certain gauge fields in an algebraic way without preserving the $SU(2, 2|N)$ invariances. Therefore the transformations of these fields will differ from (2.1) and because of that the original algebra will change. These changes are more involved than just a straightforward conversion of P into general coordinate transformations, and in the general case it is known that the algebra of gauge transformations will no longer close. This shows that the original gauge field representation is too small for

generating a supersymmetry algebra of the conventional type*. To find a complete multiplet of fields and the corresponding superconformal algebra in its linearized form is therefore the first step in the construction of superconformal gravity. This requires the use of a different method such as the one we discuss below. Notwithstanding the above complications it remains extremely useful to formulate the superconformal theory in terms of $SU(2, 2|N)$ gauge fields and transformations. However, it turns out that the P transformations will no longer play a role in the algebra that we ultimately obtain. Therefore, they will be discarded henceforth.

Before discussing a complete multiplet of fields for extended conformal supergravity we present the curvature constraints for the $SU(2, 2|N)$ gauge fields. It is known from $N = 1$ and 2 , that ω_μ^{ab} , f_μ^a and ϕ_μ^i are to be determined by constraints [1, 3]. We make the following choice:

$$\hat{R}_{\mu\nu}{}^a(P) = 0, \quad (2.3)$$

$$\hat{R}_{\mu\nu}{}^{ab}(M)e^\nu{}_b = 0, \quad (2.4)$$

$$\gamma^\mu \hat{R}_{\mu\nu}{}^i(Q) = 0. \quad (2.5)$$

Other choices of constraints, such as the ones presented for $N = 1, 2$, are related to these ones by field redefinitions. We have used the notation \hat{R} to indicate that the curvatures in (2.3)–(2.5) will contain extra covariantizations induced by modifications of the transformation laws.

The constraints (2.3)–(2.5) lead to further restrictions on the curvatures if one makes use of the $SU(2, 2|N)$ Bianchi identities. The following useful identities, which are not independent, can be derived in that way:

$$\hat{R}_{\mu\nu}(D) = 0, \quad (2.6)$$

$$\varepsilon^{aec d} \hat{R}_{cd}{}^{be}(M) = 0, \quad (2.7)$$

$$\hat{R}_{ab}{}^{cd}(M) = \hat{R}_{cd}{}^{ab}(M), \quad (2.8)$$

$$\frac{1}{4}\varepsilon_{ab}{}^{cd}\varepsilon^{ef}{}_{gh}\hat{R}_{cd}{}^{gh}(M) = \hat{R}_{ab}{}^{ef}(M), \quad (2.9)$$

$$\varepsilon^{cdef}D_b D_d \hat{R}_{ef}{}^{ab}(M) = 0, \quad (2.10)$$

$$\hat{R}_{ab}{}^c(K) = D_e \hat{R}_{ab}{}^{ec}(M), \quad (2.11)$$

$$\varepsilon^{abcd}D_b \hat{R}_{cd}(A) = 0, \quad (2.12)$$

$$\varepsilon^{abcd}D_b \hat{R}_{cd}{}^i(V) = 0, \quad (2.13)$$

$$D_a \hat{R}_{ab}{}^i(Q) = -\frac{1}{4}\varepsilon^{abcd}\gamma_a \hat{R}_{cd}{}^i(S), \quad (2.14)$$

$$\hat{\hat{R}}_{ab}{}^i(Q) + \hat{R}_{ab}{}^i(Q) = 0, \quad (2.15)$$

$$\hat{R}_{ab}{}^i(S) - \hat{\hat{R}}_{ab}{}^i(S) = 2\mathcal{D}\hat{R}_{ab}{}^i(Q), \quad (2.16)$$

* We have presented a counting argument for general N in [4]. The same argument was used for $N = 1$ in the second work of [1].

$$\sigma^{ab}\hat{R}_{ab}{}^i(S) = 0, \quad (2.17)$$

$$\varepsilon^{abcd}D_b\hat{R}_{cd}{}^i(S) = 0. \quad (2.18)$$

The derivatives in (2.6)–(2.18) are covariant with respect to all the symmetries that were discussed previously (except P transformations). However, for the non-linear theory some of these identities have modifications, which we shall discuss in sect. 4.

We have argued that the field representation of the $SU(2, 2|N)$ gauge theory is too small in the presence of the curvature constraints. One way to find the complete field representation is by constructing the supermultiplet of currents for a conformally invariant matter field theory with rigid supersymmetry [10, 11]. The obvious candidate for this is the $N = 4$ Yang–Mills theory [19]; we shall discuss this theory and the corresponding multiplet of currents in appendix A. Other methods for constructing the field representation of conformal supergravity will be discussed in the next section. Conformal supergravity with $N < 4$ can be readily obtained by reduction from the $N = 4$ theory.

The multiplet of currents for $N = 4$ supersymmetric Yang–Mills theory contains the energy-momentum tensor $\theta_{\mu\nu}$, the supersymmetry currents $J_\mu{}^i$, and the chiral $SU(4)$ currents, which are the Noether currents of translations, supersymmetry transformations, and chiral $SU(4)$ transformations, respectively. These currents are bilinear in the fields, and if the latter satisfy their field equations they are conserved because of the invariance of the theory under the corresponding transformations. Supersymmetry further extends this set of currents to a full supermultiplet of bilinear operators. The relation of the multiplet of currents to the $SU(2, 2|4)$ gauge fields becomes clear when one envisages the coupling of the Yang–Mills theory to supergravity. The currents will couple to the gauge fields $e_\mu{}^a$, $\psi_\mu{}^i$ and $V_\mu{}^i$. The fields $\omega_\mu{}^{ab}$, $f_\mu{}^a$ and $\phi_\mu{}^i$ are not independent, as is expressed by the constraints (2.3)–(2.5). When this is taken into account their couplings correspond to form-conserved improvement terms that can be added to the currents to make the energy-momentum tensor symmetric and traceless ($\theta_{\mu\nu} = \theta_{\nu\mu}$; $\theta_{\mu\mu} = 0$) and the supersymmetry current traceless ($\gamma^\mu J_\mu{}^i = 0$). The invariance of the full theory under Lorentz transformations, dilatations and S supersymmetry is then reflected in the above restrictions on the currents.

From the multiplet of currents one may construct a corresponding supermultiplet of fields. One first assigns a field to each component of the current multiplet, and forms a generalized inner product of field and current components. This inner product is required to be supersymmetric, from which one derives the transformation properties of the field supermultiplet. The field supermultiplet contains the gauge fields $e_\mu{}^a$, $\psi_\mu{}^i$ and $V_\mu{}^i$, and matter fields: fermionic fields Λ_i and $\chi^\eta{}_k$, and bosonic fields C , E_η , $T_{ab}{}^\eta$ and $D^\eta{}_{kl}$. The next step is to make the other gauge fields of $SU(2, 2|N)$ explicit in the transformation rules. This is done straightforwardly by re-expressing derivatives on $e_\mu{}^a$, $\psi_\mu{}^i$ and $V_\mu{}^i$ in terms of the $SU(2, 2|N)$ curvatures, using the explicit expressions for $\omega_\mu{}^{ab}$, $f_\mu{}^a$ and $\phi_\mu{}^i$ as given by the curvature

constraints (2.3)–(2.5). At this point we find that apart from the matter fields the structure of the transformation rules agrees with (2.1). However, the dilatational gauge field b_μ is still lacking; because of conformal invariance the Yang–Mills fields do not couple to b_μ . Therefore, this field is reinstated by hand, by making derivatives covariant with respect to dilatations and by choosing its transformation rules according to (2.1). The S -supersymmetry transformations of the matter fields follow from the calculation of the $[Q, K]$ commutator, where one has to introduce supercovariant derivatives in their Q -supersymmetry transformations. In doing so one chooses these fields inert under K transformations. The consistency of this assumption is checked *a posteriori* by calculating the $[S, S]$ commutator.

We have thus found the field representation for $N = 4$ conformal supergravity with the linearized transformation laws. We give these transformations for Q and S supersymmetry:

$$\begin{aligned}
\delta C &= \bar{\epsilon}^i \Lambda_i, \\
\delta \Lambda_i &= 2\mathcal{D}C\epsilon_i + E_{ij}\epsilon^j + \epsilon_{ijkl}\sigma \cdot T^{kl}\epsilon^j, \\
\delta E_{ij} &= \bar{\epsilon}_{(i}\mathcal{D}\Lambda_{j)} - \bar{\epsilon}^k \chi^{mn}{}_{(i}\epsilon_{j)kmn} + \bar{\eta}_{(i}\Lambda_{j)}, \\
\delta T_{ab}{}^{ij} &= \bar{\epsilon}^{[i}R_{ab}{}^{j]}(Q) + \bar{\epsilon}^k \sigma_{ab}\chi^{ij}{}_k + \frac{1}{2}\epsilon^{ijkl}\bar{\epsilon}_k\mathcal{D}\sigma_{ab}\Lambda_l - \frac{1}{2}\epsilon^{ijkl}\bar{\eta}_k\sigma_{ab}\Lambda_l, \\
\delta \chi^{ij}{}_k &= -\sigma \cdot T^{ij}\bar{\mathcal{D}}\epsilon_k - \frac{1}{3}\delta^{[i}{}_k\sigma \cdot T^{j]l}\bar{\mathcal{D}}\epsilon_l - \sigma \cdot R^{[i}{}_k(V)\epsilon^{j]} - \frac{1}{3}\delta^{[i}{}_k\sigma \cdot R^{j]}{}_l(V)\epsilon^l \\
&\quad - \frac{1}{2}\epsilon^{ijlm}\mathcal{D}E_{kl}\epsilon_m + D^{ij}{}_kl\epsilon^l \\
&\quad + \sigma \cdot T^{ij}\eta_k + \frac{1}{3}\delta^{[i}{}_k\sigma \cdot T^{j]l}\eta_l - \frac{1}{2}\epsilon^{ijlm}E_{kl}\eta_m, \\
\delta D^{ij}{}_{kl} &= -2\bar{\epsilon}^{[i}\mathcal{D}\chi^{j]}{}_{kl} + \delta^{[i}{}_{[k}\bar{\epsilon}^m\mathcal{D}\chi^{j]}{}_{m]l} + \text{h.c.}, \\
\delta e_\mu{}^a &= \bar{\epsilon}^i\gamma^a\psi_{\mu i} + \text{h.c.}, \\
\delta \psi_\mu{}^i &= 2(\partial_\mu\epsilon^i + \frac{1}{2}b_\mu\epsilon^i - \frac{1}{2}\omega_\mu{}^j\sigma\epsilon^j - V_\mu{}^j\epsilon^j) - \sigma \cdot T^{ij}\gamma_\mu\epsilon_j - \gamma_\mu\eta^i, \\
\delta V_\mu{}^i &= \bar{\epsilon}^j\phi_{\mu j} + \bar{\epsilon}^k\gamma_\mu\chi^i{}_{kj} - \frac{1}{4}\delta^i{}_j\bar{\epsilon}^k\phi_{\mu k} - \bar{\psi}_\mu{}^j\eta_j + \frac{1}{4}\delta^i{}_j\bar{\psi}_\mu{}^k\eta_k - \text{h.c.}, \\
\delta b_\mu &= \frac{1}{2}\bar{\epsilon}^i\phi_{\mu i} - \frac{1}{2}\bar{\psi}_\mu{}^i\eta_i + \text{h.c.}
\end{aligned} \tag{2.19}$$

We have listed the various fields with their algebraic restrictions in table 3. Their transformations under local dilatations are characterized by a Weyl weight factor w . In addition the transformations (2.19) are compatible with a rigid chiral $U(1)$ symmetry, which is characterized by a chiral weight factor c . Both weights are included in table 3.

Hence we have found the field representation of the $N = 4$ superconformal theory. The supermultiplet contains $128 + 128$ degrees of freedom. Our construction guarantees that the superconformal gauge transformations have a closed algebra in the linear approximation. Finding the full non-linear transformations and the superconformal algebra will be the subject of sect. 4.

TABLE 3
Fields of $N = 4$ conformal supergravity

Field	Type	Restrictions	SU(4)	w	c
C	boson	complex	1	0	-2
A_i	fermion	$\gamma_5 A_i = A_i$	4	$\frac{1}{2}$	$-\frac{3}{2}$
E_{ij}	boson	$E_{ij} = E_{ji}$; complex	10	1	-1
$T_{ab}{}^{ij}$	boson	$T_{ab}{}^{ij} = -T_{ba}{}^{ij} = -T_{ab}{}^{ji}$, $\frac{1}{2}\varepsilon_{ab}{}^{cd}T_{cd}{}^{ij} = -T_{ab}{}^{ij}$	6	1	-1
$\chi^{ij}{}_k$	fermion	$\gamma_5 \chi^{ij}{}_k = \chi^{ij}{}_k$, $\chi^{ij}{}_k = -\chi^{ji}{}_k$; $\chi^{ij}{}_i = 0$	20	$\frac{3}{2}$	$-\frac{1}{2}$
$D^{ij}{}_{kl}$	boson	$D^{ij}{}_{kl} = \frac{1}{4}\varepsilon^{ijmn}\varepsilon_{klpq}D^{pq}{}_{mn}$; $D^{ij}{}_{kl} \equiv (D^{kl}{}_{ij})^* = D^{ij}{}_{kl}$, $D^{ij}{}_{kl} = -D^{ji}{}_{kl} = -D^{ij}{}_{lk}$, $D^{ij}{}_{kl} = 0$	20	2	0
$e_\mu{}^a$	boson	vierbein	1	-1	0
$\psi_\mu{}^i$	fermion	$\gamma_5 \psi_\mu{}^i = \psi_\mu{}^i$; gravitino	4	$-\frac{1}{2}$	$-\frac{1}{2}$
$V_{\mu i}$	boson	$V_{\mu i} \equiv (V_{\mu}{}^i)^* = -V_{\mu}{}^i$, $V_{\mu}{}^i = 0$; SU(4) gauge field	15	0	0
b_μ	boson	dilatational gauge field	1	0	0

We have indicated the various algebraic restrictions on the fields, their representation assignments, and Weyl and chiral weight factors.

3. Linearized Weyl multiplet and superconformal actions

Originally superconformal actions have been obtained by using the geometric approach outlined in [20]. This program has been carried through completely for $N = 1$ [1], and for arbitrary N there have been some attempts to exhibit that part of the action that follows from the $SU(2, 2|N)$ algebra [6]. With our choice of constraints the gauge-field part of the action follows rather directly. We start from an action constructed from $SU(2, 2|N)$ curvatures subject to (2.3)–(2.5). The most general action invariant under dilatations has only four terms. It has the form

$$\begin{aligned}
 \mathcal{L} = & \frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu}{}^{ab}(M)R_{\rho\sigma}{}^{cd}(M)\varepsilon_{abcd} - \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(\bar{R}^i{}_{\mu\nu}(Q)R_{\rho\sigma i}(S) - \text{h.c.}) \\
 & + eR_{\mu\nu}{}^i{}_j(V)R^{\mu\nu}{}_i{}^j(V) - \frac{4-N}{4N}eR_{\mu\nu}(A)R^{\mu\nu}(A),
 \end{aligned} \quad (3.1)$$

where we have chosen relative coefficients in such a way that this lagrangian is invariant under S supersymmetry. To show this one must make use of the constraints (2.3)–(2.5). Although we do not have the same constraints, eq. (3.1) is still in agreement with the results of [6]. Notice that the coefficient of the term quadratic in $R_{\mu\nu}(A)$ changes sign for $N = 4$, a fact that we have alluded to in sect. 1.

However, beyond $N = 1$ it is difficult to complete this program because the superconformal theories contain matter fields. As an alternative one may consider the supermultiplet of gauge covariant components of the superconformal field

representations [13]. For $N \leq 4$ this multiplet can be represented by a chiral superfield, which we denote by $W(z^\mu, \theta^i)$. We will discuss the case of $N = 4$; lower N theories can be obtained by means of suitable truncations. The Weyl multiplet $W(z^\mu, \theta^i)$ is a Lorentz scalar for $N = 4$; it is a generalization of the Weyl multiplets that are known for lower N , where W is a Lorentz tensor.

Apart from the chiral constraint

$$D^k W(z^\mu, \theta^i) = 0, \quad (3.2)$$

with

$$D^i \equiv \frac{\partial}{\partial \theta_i} + \gamma^\mu \theta^i \frac{\partial}{\partial x^\mu}, \quad (3.3)$$

$$z^\mu = x^\mu + \bar{\theta}^i \gamma^\mu \theta_i,$$

one must impose an additional constraint:

$$\bar{D}_i \sigma_{ab} D_j \bar{D}_k \sigma^{ab} D_l W = \frac{1}{4} \varepsilon_{ijmn} \varepsilon_{klpq} \bar{D}^m \sigma_{ab} D^n \bar{D}^p \sigma^{ab} D^q W^*. \quad (3.4)$$

We have used the same chiral notations as in [17]: fermionic superspace coordinates θ^i and covariant derivatives D^i transform according to the defining representation of $SU(4)$, and have positive and negative chirality, respectively; the corresponding opposite-chirality components have a lower $SU(4)$ index indicating that they transform according to the conjugate representation. We recall that θ^i and D_i have chiral and dilatational weight factors $c = w = -\frac{1}{2}$ and $c = w = +\frac{1}{2}$, respectively.

We will now exhibit the relation between the superfield constraints (3.2)–(3.4) and the restrictions induced by the curvature constraints and $SU(2, 2|N)$ Bianchi identities (2.3)–(2.18). The superfield $W(z^\mu, \theta^i)$ is decomposed as follows:

$$\begin{aligned} W(z^\mu, \theta^i) = & C + \bar{\theta}^i A_i + \frac{1}{2} \bar{\theta}^i \theta^j E_{ij} + \frac{1}{2} \bar{\theta}^i \sigma_{ab} \theta^j T_{ab}{}^{kl} \varepsilon_{ijkl} \\ & + \frac{1}{3} \bar{\theta}^i \sigma_{ab} \theta^j \varepsilon_{ijkl} (\bar{\theta}^m \sigma_{ab} \chi^{kl}{}_m + \bar{\theta}^k \hat{R}^l{}_{ab}(Q) \\ & - \frac{3}{2} \bar{\theta}^k \theta^m \hat{R}_{ab}{}^l{}_m(V) + \frac{1}{4} \bar{\theta}^m \sigma_{ab} \theta^n D^{kl}{}_{mn} - \frac{1}{4} \bar{\theta}^k \sigma_{cd} \theta^l \hat{R}_{ab}{}^{cd}(M)) \\ & - \frac{1}{6} \bar{\theta}^i \sigma_{ab} \theta^j (\bar{\theta}^k \sigma_{ab} \theta^l \bar{\theta}^m \mathcal{D} \chi^{nkl} \varepsilon_{ijmn} - \frac{2}{5} \bar{\theta}^k \sigma_{cd} \theta^l \bar{\theta}^m \sigma_{ab} \hat{R}_{cd}(S) \varepsilon_{ijkl}) \\ & - \varepsilon_{ijkl} \bar{\theta}^i \sigma_{ab} \theta^j \bar{\theta}^m \sigma_{ab} \theta^n (\frac{1}{72} \bar{\theta}^k \theta^p \square E^{lq} \varepsilon_{mnpq} + \frac{1}{5} \bar{\theta}^k \sigma_{cd} \theta^l D_c D_e T_{edmn}) \\ & + \frac{1}{180} \varepsilon_{ijkl} \bar{\theta}^i \sigma_{ab} \theta^j \bar{\theta}^k \sigma_{cd} \theta^l \bar{\theta}^m \sigma_{ab} \theta^n \varepsilon_{mnpq} (\bar{\theta}^p \sigma_{cd} \square \mathcal{D} \Lambda^q + \frac{1}{4} \bar{\theta}^p \sigma_{cd} \theta^q \square^2 C^*). \end{aligned} \quad (3.5)$$

This decomposition can be found by straightforward manipulation of the θ -spinors, and subsequent identification with the components of the superconformal theory. We note some particularly useful identities:

$$\begin{aligned} \varepsilon_{ijkl} \bar{\theta}^i \sigma_{ab} \theta^j \bar{\theta}^k \sigma_{ac} &= \frac{1}{2} \varepsilon_{ijkl} \bar{\theta}^i \sigma_{bc} \theta^j \bar{\theta}^k, \\ \varepsilon_{ijkl} \bar{\theta}^i \sigma_{ab} \theta^j \bar{\theta}^k \sigma_{cd} \theta^l \bar{\theta}^n &= \varepsilon_{ijkl} \bar{\theta}^i \sigma_{ab} \theta^j \bar{\theta}^k \sigma_{ef} \theta^l \bar{\theta}^n \sigma_{cd} \sigma_{ef} \\ &\quad + \varepsilon_{ijkl} \bar{\theta}^i \sigma_{ef} \theta^j \bar{\theta}^k \sigma_{cd} \theta^l \bar{\theta}^n \sigma_{ab} \sigma_{ef}, \\ \varepsilon_{ijkl} \bar{\theta}^i \sigma_{ab} \theta^j \bar{\theta}^k \sigma_{cd} \theta^l \bar{\theta}^m \theta^n &= 0, \end{aligned} \quad (3.6)$$

which follow from Fierz re-orderings and use of Schouten identities. It is easy to verify that W represents indeed all components of the superconformal theory, in spite of the fact that (3.5) projects out certain duality and chirality combinations. Obviously W contains the matter fields C, A, E, T, χ and D , but the decomposition (3.5) only implies the presence of the traceless parts of χ and D . This is in agreement with the multiplet components introduced in the previous section (see table 3). Furthermore W only contains the antidual components of $\hat{R}(Q)$ and $\hat{R}(S)$ with positive chirality. In addition these tensors are traceless with respect to contractions with γ -matrices. A similar situation arises for $\hat{R}(M)$, of which only those components appear in (3.5) that are symmetric in index pairs, traceless, and antidual in each index pair separately. Again this agrees with the results of the previous section, where the algebraic identities of eqs. (2.3)–(2.18) show that the remaining components either vanish or are expressible in terms of other quantities. Finally the components of the superconformal multiplet which do not explicitly occur in (3.5) are expressible in terms of those which do, by virtue of the same curvature identities (2.3)–(2.18).

Hence $W(z^\mu, \theta^i)$ represents fully the superconformal theory of sect. 2. It is obvious from (3.5) that W is not an unrestricted chiral superfield. This is manifest in some of the higher- θ components which are expressed in terms of derivatives of matter fields whose complex conjugates occur in the lower- θ sector. These restrictions are due to the constraint (3.4), which also imposes further conditions on the field D^u_{kl} and the curvatures \hat{R} . An explicit evaluation of (3.4) shows that D must be real:

$$D_{ij}{}^{kl} \equiv (D^u_{kl})^* = \frac{1}{4} \varepsilon_{ijmn} \varepsilon^{klpq} D^{mn}{}_{pq} = D^{kl}{}_{ij}, \quad (3.7)$$

and that the curvatures satisfy certain differential identities. These identities and (3.7) have indeed their counterpart in the superconformal multiplet introduced in sect. 2: the field D satisfies the same reality condition (see table 3), and the curvature identities are equivalent to those given in eqs. (2.6)–(2.18). These arguments show that an explicit construction of the superfield W subject to the constraints (3.2)–(3.4) is an alternative method for finding the field representation of conformal supergravity.

So far our discussion of superfields and corresponding multiplets has been entirely within the context of rigid superspace. Although we do not intend to give a full superspace treatment of conformal supergravity here, there is one aspect that should be pointed out. It is known that the geometry of superspace can be such that certain multiplets of rigid supersymmetry will no longer exist*. For extended supersymmetry with $N > 2$ this is the case for chiral superfields, as we can already derive from the linearized results (2.19). Namely, if one computes the commutator of two Q -supersymmetry transformations one finds another Q transformation of the form

$$[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] \rightarrow \delta_Q(\varepsilon_3 = \varepsilon^{ijkl} \bar{\varepsilon}_{1k} \varepsilon_{2l} A_j). \quad (3.8)$$

* See, for instance, refs [21]

One may then try to impose such an algebra on a chiral superfield Φ , which is decomposed according to

$$\Phi = A + \bar{\theta}^i \psi_i + O(\theta^2). \quad (3.9)$$

Its lowest- θ component transforms under Q supersymmetry as

$$\delta A = \bar{\varepsilon}^i \psi_i, \quad (3.10)$$

and the commutator of two supersymmetry transformations should yield a Q transformation as specified by (3.8). However, A only transforms with supersymmetry parameters of positive chirality (this is the defining property of a chiral superfield), whereas (3.8) only contains transformation parameters ε_{1i} and ε_{2i} with *negative* chirality. Hence it is not possible to generate the right-hand side of (3.8), unless the effect on A of the Q transformation specified by (3.8) vanishes. This requires

$$\bar{\Lambda}_k \psi_l = \bar{\Lambda}_l \psi_k. \quad (3.11)$$

Eq. (3.11) is not satisfied unless ψ_i is proportional to Λ_i , which shows that chiral fields are not possible in curved superspace, with the exception of powers of the Weyl field W .

This argument can be formulated directly in terms of the torsions and curvatures of superspace. The Q transformation in (3.8) shows that the parameter of superspace-coordinate transformations Ξ^M has a term linear in θ of the form

$$\Xi^M \propto \varepsilon^{ijkl} (\bar{\theta}_j \varepsilon_k) \Lambda_{l\alpha}, \quad (3.12)$$

when M is the spinor component of the base manifold with spinor index α and upper SU(4) index i (and thus positive chirality; the opposite-chirality component is obtained by complex conjugation). From this one derives* that the supervielbein E_M^A with both M (base manifold) and A (tangent space) indices taking spinorial values α and β with corresponding upper SU(4) indices i and j , respectively, contains a term (our notation does not distinguish between M and A spinor indices):

$$E_\alpha^{i,\beta l} \propto \varepsilon^{ijkl} \bar{\theta}_{\alpha k} \Lambda_l^\beta. \quad (3.13)$$

This shows that certain of the supertorsion components with three spinorial indices are proportional to Λ . This implies that the anticommutator of two covariant derivatives has a term proportional to Λ of zeroth order in θ :

$$\{D_\alpha^i, \bar{D}_\beta^j\} \rightarrow \delta_{\alpha\beta} \varepsilon^{ijkl} \bar{\Lambda}_k D_l. \quad (3.14)$$

In other words, the torsion components are not sufficiently restricted to preserve chiral representations. When we reduce these results to $N = 3$ supergravity they agree with those of ref. [23], provided we identify Λ as the physical spinor field of

* This is done by systematically identifying the components of superfields to the results obtained in ordinary space. These superfields are thus constructed order-by-order in the Grassmann variables θ in a particular gauge. For details and further references, see, for instance, [22].

Poincaré supergravity. This identification will be motivated later in this section.

There are similar problems when chiral superfields carry Lorentz indices. The reason is that the anticommutator (3.14) has also a contribution of the supercurvature of the Lorentz transformations in the tangent space. In zeroth order of θ the curvature component with two spinorial indices is proportional to the tensor fields $T_{ab}{}^u$, and we find in addition to (3.14):

$$\{D_\alpha{}^i, \bar{D}_{\beta'}\} \rightarrow \delta_{\alpha\beta} T_{ab}{}^u \Gamma^{ab}. \quad (3.15)$$

Here Γ^{ab} denote the generators of the Lorentz transformations. This shows that chiral superfields cannot exist in arbitrary representations of the Lorentz group. This is relevant for the $N=2$ Weyl multiplet which is an antiselfdual Lorentz tensor denoted by $W_{ab}{}^u(z^\mu, \theta^k)$. In that case the chirality is preserved because Lorentz transformations of the form (3.15) vanish on $W_{ab}{}^u$. The reason is that the lowest- θ component of $W_{ab}{}^u$ is precisely equal to $T_{ab}{}^u$ itself. The case $N=2$ will be further discussed in sect. 5. We caution the reader that, although (3.14) and (3.15) represent genuine restrictions on chiral superfields, one should always consider the combined effect of all terms on the right-hand side of the anticommutator. For instance, to establish that the $N=3$ Weyl multiplet has its chirality preserved both (3.14) and (3.15) are essential. (The $N=3$ Weyl multiplet is a Lorentz spinor with Λ as its lowest- θ component.)

We now continue our discussion of the linearized superconformal theory. We remind the reader that the superconformal transformations are compatible with a rigid chiral $U(1)$ invariance. These chiral transformations as well as dilatations can be given directly for $W(z^\mu, \theta^i)$. If we denote the parameters of Weyl and chiral transformations by ζ_D and ζ_A then W transforms according to

$$W(z^\mu, \theta^i) \rightarrow e^{-2i\zeta_A} W(z^\mu, e^{-(\zeta_D + i\zeta_A)/2} \theta^i). \quad (3.16)$$

Hence W has Weyl and chiral weight factors $w=0$ and $c=2$. Notice that precisely these weights are compatible with the constraint (3.4).

It is now straightforward to find the invariant action for the linearized superconformal theory. We take the square of $W(z^\mu, \theta^i)$, which is again a chiral superfield, but now a general chiral field without restrictions such as (3.4), with Weyl and chiral weights $w=0$ and $c=4$. Its highest- θ component has weight factors $w=4$ and $c=0$, and yields a supersymmetric and chiral invariant:

$$\begin{aligned} I(W^2) = & C \square^2 C^* - \frac{1}{2} \bar{\Lambda}^i \square \not{D} \Lambda_i + 4 D_a T_{ab}{}^u D_c T_{cbu} \\ & - \frac{1}{4} D_a E^u D_a E_u - \bar{\chi}^u{}_k \not{D} \chi^k{}_u + \frac{1}{2} D^u{}_{kl} D^k{}_u + \tilde{R}_{ab}{}^i(Q) \hat{R}_{ab}{}_i(S) \\ & + \frac{1}{2} \hat{R}_{ab}{}_i(V) (\hat{R}_{ab}{}^i(V) - \frac{1}{2} \varepsilon_{abcd} \hat{R}_{cd}{}^i(V)) \\ & + \frac{1}{4} \hat{R}_{ab}{}^{cd}(M) (\hat{R}_{ab}{}^{cd}(M) - \frac{1}{2} \varepsilon_{abef} \hat{R}_{ef}{}^{cd}(M)). \end{aligned} \quad (3.17)$$

Its real part is the invariant action for $N=4$ conformal supergravity, which is an

extension of the lagrangian (3.1). The imaginary part is a total divergence. Contrary to (3.1) the result (3.17) is not invariant under S supersymmetry, not even the part consisting of squares of curvatures. The reason is that the curvatures occurring in W have extra covariantizations, which lead to modified transformation rules under supersymmetry. Of course, the variation of higher-order terms in the full nonlinear density will cancel the S variations of (3.17).

It should be possible to find the superconformal theory from an off-shell formulation of Poincaré supergravity. This is done by completing the highest-spin components of the latter theory into a supermultiplet, thereby introducing extra gauge invariances to remove some of the low-spin components. In this way the $N = 2$ superconformal theory has been constructed originally [3]. One may then apply the off-shell counting arguments of ref. [7]. Namely, one takes the sum of the linearized Weyl and Poincaré supergravity lagrangians:

$$\mathcal{L} = \mathcal{L}_W + m^2 \mathcal{L}_P, \quad (3.18)$$

with m^2 some arbitrary dimensionful constant. This lagrangian describes massive and massless states which should combine into supermultiplets. The massless supermultiplet coincides with that of the Poincaré theory.

However, for $N > 2$ no off-shell formulations of Poincaré supergravity exist to which these arguments can be applied. But on the other hand the fact that (3.18) must generate a massive supermultiplet may provide some information about the $N = 4$ Poincaré theory. Therefore we first discuss the content of the $N = 4$ massive supermultiplet. It is known [24] that massive supermultiplets can be classified according to antisymmetric tensor representations of $\text{Sp}(2N)$. In the context of the conformal theory it is desirable to decompose these representations according to the chiral $\text{SU}(N)$ subgroup of $\text{Sp}(2N)$. A general discussion of these invariance properties is given in appendix B. Application of these arguments leads directly to the $\text{SU}(4)$ decomposition of the $N = 4$ massive supermultiplet presented in table 4.

We may now compare the $\text{SU}(N)$ assignments of the spin states of table 4 with those of the superconformal fields listed in table 3. There is a perfect match: the spin-2 state and the vierbein field are both singlets, the spin- $\frac{3}{2}$ state and the gravitino fields are in quartets, the spin-1 states are in the same representations as the chiral

TABLE 4
 $\text{Sp}(8)$ and $\text{SU}(4)$ decomposition of the massive $N = 4$
 supermultiplet

Spin	$\text{Sp}(8)$	$\text{SU}(4)$
2	1	1
$\frac{3}{2}$	8	$4 + 4^c$
1	27	$15 + 6 + 6^c$
$\frac{1}{2}$	48	$20 + 20^c + 4 + 4^c$
0	42	$20 + 10 + 10^c + 1 + 1^c$

$SU(N)$ gauge fields and the tensors $T_{ab}{}^{\eta}$, and also the representations of the spin- $\frac{1}{2}$ states and of the fields χ and Λ coincide, as well as those of the spin-0 states and the fields D , E and C . Clearly these fields are directly responsible for generating the massive multiplet and it is not difficult to envisage specific mechanisms for generating these states in direct analogy to what is known to happen in $N = 2$ supergravity. We have already discussed this elsewhere [4]. Here we only repeat that in order to generate the singlet spin-0 and quartet spin- $\frac{1}{2}$ states, the corresponding fields C and Λ are expected to occur in the linearized Poincaré supergravity lagrangian according to

$$\mathcal{L} \rightarrow -|\partial_\mu C|^2 - \frac{1}{4} \bar{\Lambda} \overleftrightarrow{\not{D}} \Lambda. \quad (3.19)$$

This is also expected on dimensional grounds. In combination with the C - and Λ -dependent terms in the conformal lagrangian \mathcal{L}_w [which follows directly from eq. (3.17)] this generates indeed the required states of the massive and massless $N = 4$ supermultiplets. This analysis implies that C and Λ should be identified as the physical spin-0 and spin- $\frac{1}{2}$ fields of $N = 4$ Poincaré supergravity, as is already clear in eq. (3.19).

4. Transformations of $N = 4$ conformal supergravity

The derivation of the complete transformation rules and the corresponding superconformal algebra proceeds by means of an iterative method. One calculates the commutator algebra on the basis of the linearized transformations (2.19), and then imposes this algebra on the fields. The latter requires the addition of terms of higher order in the fields to the transformations, which introduces corresponding terms in the algebra. The results of this constructive procedure are rather complicated. Due to the fact that fields of low Weyl weight are present, many terms are possible, and most of them do indeed appear. The situation will improve if one could use an alternative formulation with a higher degree of invariance. Indeed we shall show that there exists a version of $N = 4$ conformal supergravity which has a manifest rigid $SU(1, 1)$ and an extra local chiral $U(1)$ invariance. This should not come as a surprise since the $N = 4$ Poincaré theory is known to have this invariance [25, 26]. And our first attempts to determine the complete transformation rules showed indeed indications for the presence of these symmetries here.

In this version of the theory the scalar field C corresponds to a certain parametrization of the coset space $SU(1, 1)/U(1)$. For Poincaré supergravity such coset spaces have been discussed extensively in [26], and we refer in particular to appendix A of this work where part of $N = 4$ supergravity is worked out. Here we only give the construction. We replace C by a doublet of complex fields $\Phi = (\Phi_1, \Phi_2)$, which transforms under rigid $SU(1, 1)$, and under chiral $U(1)$ with weight factor $c = -1$. This is the same $U(1)$ group that we have introduced previously (see table 3

for the chiral weights of the various fields), but now extended to a *local* invariance group. The field Φ satisfies the $SU(1, 1) \times U(1)$ invariant constraint*

$$\Phi^\alpha \Phi_\alpha = 1. \quad (4.1)$$

Because of this constraint we can assign an $SU(1, 1)$ transformation to Φ according to

$$U = \begin{pmatrix} \Phi_1 & -\Phi^2 \\ \Phi_2 & \Phi^1 \end{pmatrix}. \quad (4.2)$$

This makes the relation clear with the coset space $SU(1, 1)/U(1)$.

The formulation of the previous sections is recovered after a $U(1)$ gauge choice. For instance we may impose a reality condition on Φ_1 :

$$\Phi_1 = \Phi^1, \quad (4.3)$$

and choose the parametrization

$$\begin{aligned} \Phi_1 &= \frac{1}{\sqrt{1-|C|^2}}, \\ \Phi_2 &= \frac{C}{\sqrt{1-|C|^2}}. \end{aligned} \quad (4.4)$$

Both $U(1)$ and $SU(1, 1)$ are broken by (4.3) and we are only left with a rigid $U(1)$ symmetry. This symmetry consists of the previous $U(1)$, but now restricted to space-time-independent transformations, combined with the diagonal subgroup of $SU(1, 1)$ in such a way that (4.3) remains unaffected. This leads precisely to the chiral weight factors given in table 3.

The $SU(1, 1) \times U(1)$ invariant formulation offers important advantages. Clearly $SU(1, 1)$ invariance prevents non-polynomial modifications, since all invariants constructed from Φ are equal to constants. In terms of other fields such modifications were already excluded because of positive Weyl weights (some of the gauge fields have negative Weyl weights but their presence is already restricted by corresponding gauge invariances). Of course, we should include derivatives on Φ as well, but $D_\alpha \Phi$ has positive Weyl weight ($w = 1$). Hence the completion of the algebra and transformation rules will require only a few iterations. In fact the extra symmetries already determine all the generic terms in the transformations of Φ and Λ . One can easily establish that these fields should transform under Q supersymmetry as

$$\begin{aligned} \delta \Phi_\alpha &= -\bar{\varepsilon}^i \Lambda_i \varepsilon_{\alpha\beta} \Phi^\beta, & \delta \Phi^\alpha &= \bar{\varepsilon}_i \Lambda^i \varepsilon^{\alpha\beta} \Phi_\beta, \\ \delta \Lambda_i &= 2\varepsilon^{\alpha\beta} \Phi_\alpha \mathcal{D} \Phi_\beta \varepsilon_i + E_{ij} \varepsilon^j + \varepsilon_{ijk} \sigma \cdot T^{kl} \varepsilon^j, \end{aligned} \quad (4.5)$$

* $SU(1, 1)$ is the group of 2×2 matrices, that leave the metric $\eta = \text{diag}(1, -1)$ invariant and that have unit determinant. Beside the metric, the Levi-Civita tensor $\varepsilon_{\alpha\beta}$ is invariant. Elements of $SU(1, 1)$ satisfy $U^{-1} = \eta U^\dagger \eta$. For doublets Φ we use the notation

$$\Phi^\alpha \equiv \eta^{\alpha\beta} (\Phi_\beta)^* = (\Phi_1^*, -\Phi_2^*).$$

where we have used the covariant bilinear expression

$$\varepsilon^{\alpha\beta} \Phi_\alpha D_\alpha \Phi_\beta = \varepsilon^{\alpha\beta} \Phi_\alpha \partial_\alpha \Phi_\beta - \frac{1}{2} \bar{\psi}_\alpha{}^i \Lambda_i. \quad (4.6)$$

Under Q and S supersymmetry (4.6) transforms as

$$\delta(\varepsilon^{\alpha\beta} \Phi_\alpha D_\alpha \Phi_\beta) = \bar{\varepsilon}^i D_\alpha \Lambda_i + \frac{1}{2} \bar{\Lambda}_i \sigma \cdot T^i \gamma_a \varepsilon_i + \frac{1}{2} \bar{\varepsilon}^i \Lambda_i \bar{\Lambda}^j \gamma_a \Lambda_j - \frac{1}{2} \bar{\eta}^i \gamma_a \Lambda_i, \quad (4.7)$$

where the derivatives are now also covariant with respect to local $U(1)$ transformations (notice that this covariantization cancels in the definition (4.6)). The gauge field of $U(1)$ is not an independent field, and is given by

$$a_\mu = -\frac{1}{2} \Phi^\alpha \tilde{\partial}_\mu \Phi_\alpha - \frac{1}{2} \bar{\Lambda}^i \gamma_\mu \Lambda_i. \quad (4.8)$$

This gauge field has the following variations:

$$\begin{aligned} \delta_{U(1)} a_\mu &= i \partial_\mu \Lambda, \\ \delta_Q a_\mu &= \frac{1}{2} \bar{\varepsilon}_i \gamma_\mu \varepsilon^{\alpha\beta} \Phi_\alpha \mathcal{D} \Phi_\beta \Lambda^i - \frac{1}{4} \bar{\Lambda}^i \gamma_\mu \varepsilon^j E_{ij} \\ &\quad - \frac{1}{4} \varepsilon_{ijkl} \bar{\Lambda}^i \gamma_\mu \sigma \cdot T^{kl} \varepsilon^j + \frac{1}{4} (\bar{\Lambda}^i \gamma^a \Lambda_i - \delta^i_j \bar{\Lambda}^k \gamma^a \Lambda_k) \bar{\varepsilon}_i \gamma_a \psi_\mu{}^j - \text{h.c.}, \\ \delta_S a_\mu &= 0. \end{aligned} \quad (4.9)$$

The iterative procedure described above leads to the following Q -supersymmetry transformations:

$$\begin{aligned} \delta_Q \Phi_\alpha &= -\bar{\varepsilon}^i \Lambda_i \varepsilon_{\alpha\beta} \Phi^\beta, \\ \delta_Q \Lambda_i &= 2 \varepsilon^{\alpha\beta} \Phi_\alpha \mathcal{D} \Phi_\beta \varepsilon_i + E_{ij} \varepsilon^j + \varepsilon_{ijkl} \sigma \cdot T^{kl} \varepsilon^i, \\ \delta_Q E_{ij} &= \bar{\varepsilon}_i (\mathcal{D} \Lambda_j) - \bar{\varepsilon}^k \chi^{mn}{}_{(i} \varepsilon_{j)kmn} - \bar{\Lambda}_i \Lambda_j \bar{\varepsilon}_k \Lambda^k + \bar{\Lambda}_k \Lambda_{(i} \bar{\varepsilon}_{j)} \Lambda^k, \\ \delta_Q T_{ab}{}^{ij} &= \bar{\varepsilon}^{[i} \hat{R}_{ab}{}^{j]}(Q) + \bar{\varepsilon}^k \sigma_{ab} \chi_k{}^{ij} + \frac{1}{2} \varepsilon^{ijkl} \bar{\varepsilon}_k \mathcal{D} \sigma_{ab} \Lambda_l \\ &\quad - \frac{1}{6} E^{[ik} \bar{\varepsilon}^{j]} \sigma_{ab} \Lambda_k - \frac{1}{3} \varepsilon^{\alpha\beta} \bar{\varepsilon}^{[i} \sigma_{ab} \Phi_\alpha \mathcal{D} \Phi_\beta \Lambda^{j]}, \\ \delta_Q \chi_k{}^i &= -\sigma \cdot T^i \tilde{\mathcal{D}} \varepsilon_k - \sigma \cdot \hat{R}^{[i}{}_k(V) \varepsilon^{j]} - \frac{1}{2} \varepsilon^{ijlm} \mathcal{D} E_{kl} \varepsilon_m \\ &\quad + D^j{}_{kl} \varepsilon^i - \frac{1}{6} \varepsilon_{klmn} E^{[i}{}_{j]} (\sigma \cdot T^j) \varepsilon^{mn} + \sigma \cdot T^{mn} \varepsilon^{ij}] \\ &\quad + \frac{1}{4} E_{kl} E^{[i}{}_{j]} \varepsilon^{lm} + \varepsilon^{ijlm} \varepsilon^{\alpha\beta} \Phi_\alpha \mathcal{D} \Phi_\beta \sigma \cdot T_{kl} \varepsilon_m \\ &\quad + \gamma^a \varepsilon_n (\frac{1}{2} \varepsilon^{ijln} \bar{\chi}^m{}_{lk} - \frac{1}{4} \varepsilon^{ijlm} \bar{\chi}^n{}_{lk}) \gamma_a \Lambda_m \\ &\quad + \varepsilon^{[i} (\frac{1}{4} \bar{\Lambda}^{j]} \mathcal{D} \Lambda_k + \frac{1}{8} \bar{\Lambda}_k \mathcal{D} \Lambda^{j]}) \\ &\quad - \sigma_{ab} \varepsilon^{[i} (\frac{1}{2} \bar{\Lambda}^{j]} \gamma_a D_b \Lambda_k - \frac{1}{4} \bar{\Lambda}_k \gamma_a D_b \Lambda^{j]}) \\ &\quad - \frac{1}{12} \varepsilon^{ijlm} \Lambda_m \{ 5 \bar{\varepsilon}_l (E_{kn} \Lambda^n + 2 \varepsilon_{\alpha\beta} \Phi^\alpha \mathcal{D} \Phi^\beta \Lambda_k) - \bar{\varepsilon}_k (E_{ln} \Lambda^n + 2 \varepsilon_{\alpha\beta} \Phi^\alpha \mathcal{D} \Phi^\beta \Lambda_l) \} \\ &\quad - \frac{1}{2} \sigma \cdot T^i \gamma_a \varepsilon_{[k} \bar{\Lambda}^l \gamma_a \Lambda_l] + \frac{1}{4} \sigma \cdot T^{[il} \gamma_a \varepsilon_{k} \bar{\Lambda}^{j]} \gamma_a \Lambda_l] \\ &\quad + \frac{1}{4} \varepsilon^{[i} \bar{\Lambda}^{j]} \Lambda^m \bar{\Lambda}_k \Lambda_m - (\text{traces}), \end{aligned} \quad (4.10)$$

$$\begin{aligned}
\delta_Q D^{\prime\prime}_{kl} = & -2\bar{\varepsilon}^{[l} \mathcal{D} \chi^{j]}_{kl} + \varepsilon_{klmn} \bar{\varepsilon}^{[l} \{-E^{j]p} \chi^{mn}_{p} + \tfrac{1}{2} \sigma \cdot T^{mn} \tilde{\mathcal{D}} \Lambda^{j]}\} \\
& + \tfrac{1}{6} E^{j]m} E^{np} \Lambda_p + \tfrac{1}{3} \varepsilon^{\alpha\beta} \Phi_\alpha \mathcal{D} \Phi_\beta \Lambda^m E^{j]n} + \tfrac{1}{2} \sigma \cdot T^{mn} \Lambda_p \bar{\Lambda}^{j]p} \Lambda^p \} \\
& + \bar{\varepsilon}^{[l} \{\gamma_\alpha \chi^{m}_{kl} \bar{\Lambda}^{j]}\} \gamma_\alpha \Lambda_m - 2\varepsilon^{\alpha\beta} \Phi_\alpha \mathcal{D} \Phi_\beta \sigma \cdot T_{kl} \Lambda^{j]} \\
& + \tfrac{1}{6} \Lambda_{[k} E_{l]m} \bar{\Lambda}^{j]} \Lambda^m + \tfrac{1}{3} \sigma_{ab} \varepsilon_{\alpha\beta} \Phi^\alpha \mathcal{D} \Phi^\beta \Lambda^{j]} \bar{\Lambda}_k \sigma_{ab} \Lambda_l \} \\
& + \varepsilon^{ijmn} \bar{\varepsilon}^p T_{abkl} (2T_{abnp} \Lambda_m + T_{abmn} \Lambda_p) + (\text{h.c.}; \text{traceless}), \\
\delta_Q e_\mu^a = & \bar{\varepsilon}^i \gamma_a \psi_{\mu i} + \text{h.c.}, \\
\delta_Q \psi_\mu^i = & 2\mathcal{D}_\mu \varepsilon^i - \sigma \cdot T^{ij} \gamma_\mu \varepsilon_j - \varepsilon^{ijkl} \bar{\varepsilon}_j \psi_{\mu k} \Lambda_l, \\
\delta_Q b_\mu = & \tfrac{1}{2} \bar{\varepsilon}^i \phi_{\mu i} + \text{h.c.}, \\
\delta_Q V_{\mu j}^i = & \bar{\varepsilon}^i \phi_{\mu j} + \bar{\varepsilon}^k \gamma_\mu \chi^i_{kj} - \tfrac{1}{2} E^{ik} \varepsilon_{jkmn} \bar{\varepsilon}^m \psi_\mu^n - \tfrac{1}{6} E^{ik} \bar{\varepsilon}_j \gamma_\mu \Lambda_k \\
& + \tfrac{1}{2} \varepsilon^{iklm} \bar{\varepsilon}_k \sigma \cdot T_l \gamma_\mu \Lambda_m + \tfrac{1}{3} \varepsilon_{\alpha\beta} \bar{\varepsilon}^i \gamma_\mu \Phi^\alpha \mathcal{D} \Phi^\beta \Lambda_j \\
& - \tfrac{1}{4} \varepsilon^{iklp} \varepsilon_{lmnp} \bar{\varepsilon}^m \gamma_\mu \psi_{\mu k} \bar{\Lambda}_l \gamma_a \Lambda^n - (\text{h.c.}; \text{traceless}),
\end{aligned}$$

where ε^i is the transformation parameter of Q supersymmetry. Under S -supersymmetry transformations with parameter η^i the fields transform as follows:

$$\begin{aligned}
\delta_S \Phi_\alpha &= 0, \\
\delta_S \Lambda_i &= 0, \\
\delta_S E_{ij} &= \bar{\eta}_{(i} \Lambda_{j)}, \\
\delta_S T_{ab}^{\prime\prime} &= -\tfrac{1}{2} \varepsilon^{ijkl} \bar{\eta}_k \sigma_{ab} \Lambda_l, \\
\delta_S \chi^i_k &= \sigma \cdot T^{ij} \eta_k + \tfrac{1}{3} \delta^{[i}_k \sigma \cdot T^{j]l} \eta_l - \tfrac{1}{2} \varepsilon^{ijlm} E_{kl} \eta_m \\
& - \tfrac{1}{8} \bar{\Lambda}_k \gamma_a \Lambda^{[i} \gamma_a \eta^{j]} + \tfrac{1}{24} \delta^{[i}_k (\bar{\Lambda}_l \gamma_a \Lambda^l \gamma_a \eta^{j]} - \bar{\Lambda}_l \gamma_a \Lambda^{j]} \gamma_a \eta^i), \\
\delta_S D^{\prime\prime}_{kl} &= 0, \\
\delta_S e_{\mu a} &= 0, \\
\delta_S \psi_\mu^i &= -\gamma_\mu \eta^i, \\
\delta_S b_\mu &= -\tfrac{1}{2} \bar{\psi}_\mu^i \eta_i + \text{h.c.}, \\
\delta_S V_{\mu j}^i &= -(\bar{\psi}_\mu^i \eta_j - \tfrac{1}{4} \delta^i_j \bar{\psi}_\mu^k \eta_k) - \text{h.c.}
\end{aligned} \tag{4.11}$$

As we have described in sect. 2, the fields ω_μ^{ab} , f_μ^a and ϕ_μ^i are not independent but are determined by the curvature constraints (2.3)–(2.5). The explicit form and the corresponding transformation rules of these gauge fields have been obtained as part of the iterative procedure that led to eqs. (4.10)–(4.11). There is no need to modify the Q and S transformations of b_μ ; possible modifications correspond to field-dependent K transformations. The transformation rules of ω_μ^{ab} , f_μ^a and ϕ_μ^i under Q

and S supersymmetry are as follows:

$$\begin{aligned}
\delta_Q \omega_\mu^{ab} &= -\bar{\varepsilon}^i \sigma^{ab} \phi_{\mu i} + \bar{\varepsilon}^i \gamma_\mu \hat{R}_{abi}(Q) - 2\bar{\varepsilon}^i T_{abij} \psi_\mu^j + \text{h.c.} , \\
\delta_Q f_\mu^a &= \frac{1}{4} \bar{\varepsilon}_i \hat{R}_{\mu a}^i(S) + \bar{\varepsilon}_i \gamma_\mu D_b \hat{R}_{ba}^i(Q) \\
&\quad + 2\bar{\varepsilon}_i T_{\mu b}^i \hat{R}_{ba j}(Q) + \text{h.c.} + \text{explicit } \psi_\mu \text{ terms} , \\
\delta_Q \phi_\mu^i &= -2f_\mu^a \gamma_a \varepsilon^i + \sigma \cdot T_{ik} \gamma_\mu \sigma \cdot T^{ij} \varepsilon^k + (\frac{1}{3} \gamma_\mu \sigma_{ab} - \sigma_{ab} \gamma_\mu) \\
&\quad \times (\hat{R}_{ab}^i(V) \varepsilon^j - D_a \Phi^\alpha D_b \Phi_\alpha \varepsilon^i + \frac{1}{4} (\bar{L}^j \gamma_a D_b \Lambda_j - \text{h.c.}) \varepsilon^i) \\
&\quad + \sigma \cdot T^{ij} \tilde{D}_a \gamma_b \varepsilon_j + \text{explicit } \psi_\mu \text{ terms} , \\
\delta_S \omega_\mu^{ab} &= \bar{\psi}_\mu^i \sigma^{ab} \eta_i + \text{h.c.} , \\
\delta_S f_\mu^a &= \frac{1}{2} \bar{\eta}_i \gamma^a \phi_\mu^i - \frac{1}{4} \bar{\eta}_i \hat{R}_{\mu a}^i(Q) + \frac{1}{6} \bar{\eta}_i \sigma \cdot T^{ij} \gamma^a \psi_{\mu j} + \text{h.c.} , \\
\delta_S \phi_\mu^i &= 2\mathcal{D}_\mu \eta^i - \frac{1}{3} \gamma_\mu \sigma \cdot T^{ij} \eta_j + \frac{1}{2} \varepsilon^{ijkl} \bar{\eta}_k \Lambda_l \psi_{\mu j} .
\end{aligned} \tag{4.12}$$

The results (4.10)–(4.12) can now be used to construct the covariant curvatures of the superconformal theory. These curvatures differ from those of the $SU(2, 2|4)$ algebra, listed in table 2. The differences are given in table 5. To determine the transformation rules of the covariant curvatures is rather straightforward, and we give some of them below.

$$\begin{aligned}
\delta_Q \hat{R}_{ab}^{cd}(M) &= -\frac{1}{4} \bar{\varepsilon}^i \{ \sigma_{ab} (\hat{R}(S) - \hat{R}(S))_{cdi} + \sigma_{cd} (\hat{R}(S) - \hat{R}(S))_{abi} \} \\
&\quad + \frac{1}{2} \bar{\varepsilon}^i \mathcal{D}(\sigma_{ab} \hat{R}_{cdi}(Q) + \sigma_{cd} \hat{R}_{abi}(Q)) + \text{h.c.} , \\
\delta_Q \hat{R}_{ab}^i(Q) &= -\hat{R}_{ab}^{cd}(M) \sigma_{cd} \varepsilon^i + (\sigma_{cd} \sigma_{ab} + \frac{1}{3} \sigma_{ab} \sigma_{cd}) (\hat{R}_{cd}^i(V) \varepsilon^j \\
&\quad - D_c \Phi^\alpha D_d \Phi_\alpha \varepsilon^i + \frac{1}{4} (\bar{L}^j \gamma_c D_d \Lambda_j + \text{h.c.}) \varepsilon^i + \mathcal{D} T_{cd}^j \varepsilon_j) , \\
\delta_Q \hat{R}_{ab}^i(V) &= \bar{\varepsilon}^i \hat{R}_{abj}(S) - \bar{\varepsilon}^k \gamma_{[a} D_{b]} \chi^i_{k j} + 2\bar{\varepsilon}^i \chi^i_{k j} T_{ab}^{kl} \\
&\quad + \frac{1}{3} T_{abjl} (2\varepsilon^{\alpha\beta} \bar{\varepsilon}^l \Phi_\alpha \mathcal{D} \Phi_\beta \Lambda^i - \bar{\varepsilon}^l E^{ik} \Lambda_k) \\
&\quad + \frac{1}{4} \varepsilon^{iklm} \bar{\varepsilon}^n \gamma_{[a} \sigma \cdot T_{kn} \sigma \cdot T_{lj} \gamma_{b]} \Lambda_m - \frac{1}{2} E^{ik} \varepsilon_{jkmn} \bar{\varepsilon}^m \hat{R}_{ab}^n(Q) \\
&\quad - \frac{1}{4} \varepsilon^{iklp} \varepsilon_{jmn p} \bar{\varepsilon}^m \gamma_c \hat{R}_{abk}(Q) \bar{L}_l \gamma_c \Lambda^n \\
&\quad + \frac{1}{6} \bar{\varepsilon}_j \gamma_{[a} D_{b]} (E^{ik} \Lambda_k) + \frac{1}{2} \varepsilon^{iklm} \bar{\varepsilon}_k D_{[a} (\sigma \cdot T_{lj} \gamma_{b]} \Lambda_m) \\
&\quad - \frac{1}{3} \varepsilon^{\alpha\beta} \bar{\varepsilon}_j \gamma_{[a} D_{b]} (\Phi_\alpha \mathcal{D} \Phi_\beta \Lambda^i) - (\text{h.c.}; \text{traceless}) , \\
\delta_S \hat{R}_{ab}^{cd}(M) &= -\frac{3}{2} \bar{\eta}_i (\sigma_{ab} \hat{R}_{cd}^i(Q) + \sigma_{cd} \hat{R}_{ab}^i(Q)) + \text{h.c.} , \\
\delta_S \hat{R}_{ab}^i(Q) &= 2(\sigma \cdot T^{ij} \sigma_{ab} + \frac{1}{3} \sigma_{ab} \sigma \cdot T^{ij}) \eta_j , \\
\delta_S \hat{R}_{ab}^i(V) &= \bar{\eta}^i \hat{R}_{abj}(Q) + \varepsilon^{iklm} T_{abjl} \bar{\eta}_k \Lambda_m - 2\bar{\eta}^k \sigma_{ab} \chi^i_{k j} \\
&\quad - \frac{1}{3} \bar{\eta}^i \sigma_{ab} (2\varepsilon_{\alpha\beta} \Phi^\alpha \mathcal{D} \Phi^\beta \Lambda_j - E_{jk} \Lambda^k) - (\text{h.c.}; \text{traceless}) .
\end{aligned} \tag{4.13}$$

TABLE 5

Extension of the $SU(2, 2|4)$ curvatures to the fully covariant curvatures of $N = 4$ conformal supergravity

$$\begin{aligned}
\hat{R}_{\mu\nu}{}^a(P) - R_{\mu\nu}{}^a(P) &= 0, \\
\hat{R}_{\mu\nu}{}^{ab}(M) - R_{\mu\nu}{}^{ab}(M) &= -\frac{1}{2}\bar{\psi}_{[\mu}\gamma_{\nu]}\hat{R}_{ab}{}^i(Q) + \bar{\psi}_{\mu}T_{ab}{}^i\psi_{\nu} + \text{h.c.}, \\
\hat{R}_{\mu\nu}(D) - R_{\mu\nu}(D) &= 0, \\
\hat{R}_{\mu\nu}{}^a(K) - R_{\mu\nu}{}^a(K) &= -\frac{1}{8}\bar{\phi}_{[\mu}\hat{R}_{\nu]}{}^i(Q) - \frac{1}{12}\bar{\phi}_{[\mu}\sigma \cdot T^{\nu}\gamma^a\psi_{\nu]} \\
&\quad - \frac{1}{8}\bar{\psi}_{[\mu}\hat{R}_{\nu]}{}^i(S) - \frac{1}{2}\bar{\psi}_{[\mu}\gamma_{\nu]}D_b\hat{R}_{ba}{}^i(Q) \\
&\quad - \bar{\psi}_{[\mu}T_{\nu]}{}^b\hat{R}_{ba}{}^i(Q) + \text{h.c.} + \text{explicit } \psi_{\mu}^2 \text{ terms}, \\
\hat{R}_{\mu\nu}{}^i(Q) - R_{\mu\nu}{}^i(Q) &= -\frac{1}{2}\sigma \cdot T^{\mu}\gamma_{[\mu}\psi_{\nu]} + \frac{1}{2}\varepsilon^{\mu\kappa l}\bar{\psi}_{\mu}\psi_{\nu}\Lambda_l, \\
\hat{R}_{\mu\nu}{}^i(S) - R_{\mu\nu}{}^i(S) &= -\frac{1}{6}\gamma_{[\mu}\sigma \cdot T^{\nu]}\phi_{\nu]} - \frac{1}{4}\varepsilon^{\mu\kappa l}\bar{\phi}_{[\mu}\Lambda_l\psi_{\nu]} \\
&\quad + \frac{1}{2}(\frac{1}{3}\gamma_{[\mu}\sigma^{ab} - \sigma^{ab}\gamma_{\mu]}) (\hat{R}_{ab}{}^i(V)\psi_{\nu]} - D_a\Phi^a D_b\Phi_a\psi_{\nu]} \\
&\quad + \frac{1}{4}(\bar{\Lambda}^i\gamma_a D_b\Lambda_i - \text{h.c.})\psi_{\nu]} + \sigma \cdot T^{\nu]}\bar{D}_a\gamma_b\psi_{\nu]} \\
&\quad + \frac{1}{2}\sigma \cdot T_{jk}\gamma_{[\mu}\sigma \cdot T^{\nu]}\psi_{\nu]}^k + \text{explicit } \psi_{\mu}^2 \text{ terms}, \\
\hat{R}_{\mu\nu}{}^i(V) - R_{\mu\nu}{}^i(V) &= -\frac{1}{2}\bar{\psi}_{[\mu}{}^k\gamma_{\nu]}\chi^i{}_{kj} + \frac{1}{4}E^{ik}\varepsilon_{jkmn}\bar{\psi}_{\mu}{}^m\psi_{\nu}{}^n \\
&\quad + \frac{1}{12}E^{ik}\bar{\psi}_{[\mu}\gamma_{\nu]}\Lambda_k - \frac{1}{4}\varepsilon^{iklm}\bar{\psi}_{[\mu}\sigma \cdot T_{l\nu}\gamma_{\nu]}\Lambda_m \\
&\quad - \frac{1}{6}\varepsilon_{\alpha\beta}\bar{\psi}_{[\mu}\gamma_{\nu]}\Phi^{\alpha}\mathcal{D}\Phi^{\beta}\Lambda_i + \frac{1}{8}\varepsilon^{\rho\kappa l}\varepsilon_{\rho lmn}\bar{\psi}_{\mu}{}^m\gamma^a\psi_{\nu}\bar{\Lambda}_l\gamma_a\Lambda^n - (\text{h.c. ; traceless})
\end{aligned}$$

The previous results also enable one to determine the complete form of the Bianchi identities (2.6)–(2.18). It turns out that most of them remain valid without further modifications. We find that only (2.10), (2.13) and (2.18) have extra non-linear terms. For instance, the complete version of eq. (2.13) takes the form

$$\varepsilon^{abcd}D_b\hat{R}_{cd}{}^i(V) = \frac{1}{2}\varepsilon^{iklm}\bar{\Lambda}_m\gamma^b\sigma \cdot T_{jl}\hat{R}_{abk}(Q) - (\text{h.c. ; traceless}). \quad (4.14)$$

Finally we present the main commutators of the superconformal algebra. The commutator of two S -supersymmetry transformations still coincides with the corresponding commutator of the $SU(2, 2|4)$ algebra. However, the commutator of two Q transformations, and of a Q and an S transformation have modifications in the form of field-dependent supersymmetry transformations. The results are given by

$$\begin{aligned}
[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] &= \delta_G^{\text{cov}}(\xi^{\mu}) + \delta_M(\varepsilon^{ab}) + \delta_Q(\varepsilon_3^i) + \delta_S(\eta^i) \\
&\quad + \delta_{SU(4)}(\Lambda^i) + \delta_{U(1)}(\Lambda) + \delta_K(\Lambda_K^a), \quad (4.15)
\end{aligned}$$

with δ_G^{cov} a covariant general coordinate transformation [27], and with the parameters of the transformations on the right-hand side of (4.15) equal to

$$\begin{aligned}
\xi^{\mu} &= 2\bar{\varepsilon}_1{}^i\gamma_{\mu}\varepsilon_{2i} + \text{h.c.}, \\
\varepsilon^{ab} &= 4\bar{\varepsilon}_1{}^i\varepsilon_2{}^jT_{abij} + \text{h.c.}, \\
\varepsilon_3^i &= \varepsilon^{\mu\kappa l}\bar{\varepsilon}_{1\kappa}\varepsilon_{2l}\Lambda_j, \\
\eta^i &= -2\bar{\varepsilon}_1{}^k\varepsilon_2{}^l\chi^i{}_{kl} - \frac{1}{2}(\bar{\varepsilon}_2{}^k\gamma_a\varepsilon_{1i} + \text{h.c.})(\gamma_a\chi^i{}_k + \varepsilon^{\mu lm}\sigma \cdot T_{km}\gamma_a\Lambda_l) \\
&\quad - \frac{1}{12}(\bar{\varepsilon}_2{}^i\gamma_a\varepsilon_{1i} - \delta^i{}_j\bar{\varepsilon}_2{}^j\gamma_a\varepsilon_{1i} + \text{h.c.})\gamma_a(E^{jk}\Lambda_k - 2\varepsilon^{\alpha\beta}\Phi_{\alpha}\mathcal{D}\Phi_{\beta}\Lambda^i) \\
&\quad + \frac{1}{2}\varepsilon^{\mu\kappa l}\bar{\varepsilon}_{1\kappa}\varepsilon_{2l}\mathcal{D}\Lambda_j + \frac{1}{3}\bar{\varepsilon}_2{}^{[i}\varepsilon_1{}^{j]}(E_{jk}\Lambda^k + 2\varepsilon_{\alpha\beta}\Phi^{\alpha}\mathcal{D}\Phi^{\beta}\Lambda_j), \quad (4.16)
\end{aligned}$$

$$\begin{aligned}
\Lambda^i{}_j &= E^{ik} \varepsilon_{klmj} \bar{\varepsilon}_2^l \varepsilon_1^m + \frac{1}{2} (\bar{\varepsilon}_2^k \gamma_a \varepsilon_{1j} + \text{h.c.}) \bar{\Lambda}^i \gamma_a \Lambda_k \\
&\quad - \frac{1}{4} (\bar{\varepsilon}_2^k \gamma_a \varepsilon_{1k} + \text{h.c.}) \bar{\Lambda}^i \gamma_a \Lambda_j \\
&\quad - \frac{1}{4} (\bar{\varepsilon}_2^i \gamma_a \varepsilon_{1j} + \text{h.c.}) \bar{\Lambda}^k \gamma_a \Lambda_k - (\text{h.c.}; \text{traceless}), \\
\Lambda &= -\frac{1}{2} i (\bar{\varepsilon}_2^i \gamma_a \varepsilon_{1j} + \text{h.c.}) (\bar{\Lambda}^i \gamma_a \Lambda_j - \delta^i_j \bar{\Lambda}^k \gamma_a \Lambda_k), \\
\Lambda_K^a &= \frac{2}{3} \bar{\varepsilon}_{2i} \gamma_b \varepsilon_1^i \bar{R}_{ab}{}^i{}_j (V) + \frac{8}{3} \bar{\varepsilon}_{2i} \varepsilon_{1j} D_b T_{ab}{}^j \\
&\quad - \frac{1}{3} \bar{\varepsilon}_{2i} \gamma_b \varepsilon_1^i \varepsilon^{abcd} (D_c \Phi^a D_d \Phi_a - \frac{1}{4} (\bar{\Lambda}^i \gamma_c D_d \Lambda_j - \text{h.c.})) \\
&\quad + \bar{\varepsilon}_{2i} \sigma \cdot T_{jk} \gamma^a \sigma \cdot T^j{}_i \varepsilon_1^k + \text{h.c.}
\end{aligned}$$

Furthermore we have

$$[\delta_Q(\varepsilon), \delta_S(\eta)] = \delta_D(\Lambda_D) + \delta_M(\varepsilon^{ab}) + \delta_S(\eta_2^i) + \delta_{\text{SU}(4)}(\Lambda^i{}_j) + \delta_K(\Lambda_K^a), \quad (4.17)$$

with the following transformation parameters:

$$\begin{aligned}
\Lambda_D &= -\bar{\eta}_i \varepsilon^i + \text{h.c.}, \\
\varepsilon^{ab} &= -2\bar{\eta}_i \sigma^{ab} \varepsilon^i + \text{h.c.}, \\
\Lambda^i{}_j &= -2\bar{\varepsilon}^i \eta_j + \frac{1}{2} \delta^i_j \bar{\varepsilon}^k \eta_k - \text{h.c.}, \\
\eta_2^i &= -\frac{1}{4} \varepsilon^{ijkl} \bar{\eta}_k \gamma_a \varepsilon_j \gamma_a \Lambda_l, \\
\Lambda_K^a &= \frac{1}{3} \bar{\eta}_i \sigma \cdot T^i{}_j \gamma_a \varepsilon_j + \text{h.c.}
\end{aligned} \quad (4.18)$$

5. $N = 2$ conformal supergravity

Certain aspects of the superconformal theory as described in sects. 2 and 3 can be illuminated by a detailed look at the $N = 2$ theory. The multiplet of the $N = 2$ superconformal fields was obtained from the $N = 2$ Poincaré theory in ref. [3]. For the lower- N theories it is often convenient to choose constraints on the $\text{SU}(2, 2|N)$ curvatures which differ from (2.3)–(2.5). For $N = 2$, these constraints can be chosen such that they are invariant under S supersymmetry. In that case, the constraints do not induce modifications in the S -supersymmetry transformations that follow from $\text{SU}(2, 2|2)$. This choice simplifies matters considerably, and we shall adhere to it throughout this section. The $N = 2$ constraints are

$$\begin{aligned}
\hat{R}_{\mu\nu}(P) &= 0, \\
\gamma^a \hat{R}_{ab}{}^i(Q) + \frac{1}{2} \gamma_b \chi^i &= 0, \\
\hat{R}_{\mu\nu}{}^{ab}(M) e^\nu{}_b - \frac{1}{2} i \hat{R}_{\mu a}(A) + 2 T_{abij} T_{\mu b}{}^{ij} - \frac{3}{2} e_\mu{}^a D &= 0.
\end{aligned} \quad (5.1)$$

In [3] the third constraint contained an arbitrary constant a , which we have chosen equal to one. We have redefined the fields $T_{ab}{}^{ij}$, χ^i , $V_{\mu i}$ and A_μ by convenient multiplicative factors with respect to [3] and [17], and we have deleted the subscript

C on the fields χ^i and D . Through the Bianchi identities the constraints lead to further restrictions on the curvatures. These differ somewhat from (2.6)–(2.18) because of the choice of the constraints. We shall present and discuss these identities later on, when we have given the full non-linear theory.

For $N = 2$, the Weyl multiplet of conformal supergravity is a chiral multiplet with external Lorentz and $SU(2)$ indices. We can represent such a multiplet by a chiral superfield $W_{ab}{}^u$, antiself-dual in its Lorentz indices and antisymmetric in its $SU(2)$ indices. It has Weyl weight $w = 1$. This superfield, which we employ only to have a convenient notation for the components of chiral multiplets, can be expanded in powers of θ as follows:

$$W_{ab}{}^u(z, \theta) = A_{ab}{}^u + \bar{\theta}^{[i} \psi_{ab}{}^{j]} + \frac{1}{2} \bar{\theta}^k \theta^{[i} B_{ab}{}^{j]}{}_k + \bar{\theta}^i \sigma_{cd} \theta^j F_{ab}{}^{cd} \\ + \frac{1}{3} \varepsilon_{kl} \bar{\theta}^k \sigma_{cd} \theta^l \bar{\theta}^m \sigma_{cd} \chi_{abm} \varepsilon^u + \frac{1}{12} (\varepsilon_{mn} \bar{\theta}^m \sigma_{cd} \theta^n)^2 \varepsilon^u \varepsilon^{kl} C_{abkl}. \quad (5.2)$$

It is clear that this superfield is a truncation of the $N = 4$ superfield (3.5). The tensors appearing in (5.2) are all antiself-dual in their Lorentz indices ($F_{ab}{}^{cd}$ in both pairs of indices). The tensor-spinors $\psi_{ab}{}^i$ and $\chi_{ab}{}^i$ are left-handed, while $B_{ab}{}^i{}_j$ is traceless. The superfield (5.2) is again a reduced superfield. It satisfies a reality condition analogous to (3.4):

$$\bar{D}_i \sigma_{ab} D_j W_{ab}{}^u = (\bar{D}_i \sigma_{ab} D_j W_{ab}{}^u)^*. \quad (5.3)$$

The condition (5.3) implies certain relations between the components of W . These relations are

$$C_{abij} = -\partial_{[a} \partial_c A_{cb]ij} + \varepsilon_{abef} \partial_e \partial_c A_{cfij}, \\ \partial_a B_{ab}{}^i + \text{h.c.} = 0, \\ F_{ab}{}^{ab} - \text{h.c.} = 0, \\ \partial_a F_{ac}{}^{bc} - \frac{1}{4} \partial_b F_{ac}{}^{ac} + \text{h.c.} = 0, \\ \partial_c \partial_d (F_{bc}{}^{ad} + F_{ac}{}^{bd} - \text{h.c.}) = 0, \\ \sigma \cdot \chi^i + \bar{\theta} \sigma \cdot \psi^i = 0, \\ \partial_a \chi_{ac}{}^i = \partial_a \bar{\theta} \psi_{ac}{}^i.$$

Note that the tensor-spinor $\chi_{ab}{}^i$ with an upper $SU(2)$ index is right-handed, and self-dual in its Lorentz indices. The relations (5.4) ensure that the reduced superfield (5.2) contains $24 + 24$ independent fermionic and bosonic components, as does the Weyl multiplet. From the expansion (5.2) the transformation rules of the corresponding chiral multiplet under rigid supersymmetry can be easily read off.

The chiral multiplet (5.2) can be extended to a full superconformal multiplet. However, as we discussed in sect. 3, not all multiplets of rigid supersymmetry can be extended to the local theory. In this case, this forces us to identify $W_{ab}{}^u$ with the components of the Weyl multiplet (or powers of the Weyl multiplet). In this identification, the multiplet restriction (5.4) and the curvature constraints and Bianchi identities of $SU(2, 2|2)$ play a crucial role, as we have also explained in sect. 3.

The full superconformal transformations of the chiral multiplet are determined modulo field redefinitions. These transformation rules are relatively simple if we make the following choice for the components of $W_{ab}{}^{\eta}$:

$$\begin{aligned}
 A_{ab}{}^{\eta} &= T_{ab}{}^{\eta}, \\
 \psi_{ab}{}^i &= \hat{R}_{ab}{}^i(Q), \\
 B_{ab}{}^i{}_j &= \hat{R}_{ab}{}^i{}_j(V) - \frac{1}{2}\varepsilon_{abcd}\hat{R}_{cd}{}^i{}_j(V), \\
 F_{ab}{}^{cd} &= -\frac{1}{2}(\mathcal{R}_{ab}{}^{cd}(M) - \frac{1}{2}\varepsilon_{abef}\mathcal{R}_{ef}{}^{cd}(M)), \\
 \chi_{abj} &= \frac{1}{2}(\mathcal{R}_{abj}(S) - \frac{1}{2}\varepsilon_{abcd}\mathcal{R}_{cdj}(S)), \\
 C_{abij} &= -(D_{[a}D_c T_{cb]ij} - \varepsilon_{abef}D_e D_c T_{cfij}).
 \end{aligned} \tag{5.5}$$

On the right-hand side of (5.5) we have the curvatures of the complete $N=2$ superconformal theory, as presented in [3]. The curvatures $\mathcal{R}_{ab}{}^{cd}(M)$ and $\mathcal{R}_{abi}(S)$ contain non-linear modifications:

$$\mathcal{R}_{ab}{}^{cd}(M) = \hat{R}_{ab}{}^{cd}(M) + T_{abij}T_{cd}{}^{ij} + T_{ab}{}^{ij}T_{cdij}, \tag{5.6}$$

$$\mathcal{R}_{abj}(S) = \hat{R}_{abj}(S) + \frac{1}{2}\gamma_{[a}D_{b]}\chi_j + T_{abjk}\chi^k. \tag{5.7}$$

The complete transformation rules of the components of the Weyl multiplet can be readily obtained from ref. [3]. We have the following Q - and S -supersymmetry transformations:

$$\begin{aligned}
 \delta T_{ab}{}^{\eta} &= \bar{\varepsilon}^i \hat{R}_{ab}{}^i{}_{\eta}(Q), \\
 \delta \hat{R}_{ab}{}^i{}_{\eta}(Q) &= -2\mathcal{D}T_{ab}{}^{\eta}{}_{\varepsilon_j} + (\hat{R}_{ab}{}^i{}_{\eta}(V) - \text{dual})\varepsilon^j \\
 &\quad - \frac{1}{2}(\mathcal{R}_{ab}{}^{cd}(M) - \frac{1}{2}\varepsilon_{abef}\mathcal{R}_{ef}{}^{cd}(M))\sigma_{cd}\varepsilon^i + 2\sigma \cdot T^{\eta}{}_{\sigma ab}\eta_j, \\
 \delta(\hat{R}_{ab}{}^i{}_{\eta}(V) - \text{dual}) &= 2\bar{\varepsilon}_j \mathcal{D}\hat{R}_{ab}{}^i{}_{\eta}(Q) - \bar{\varepsilon}^i(\mathcal{R}_{abj}(S) - \text{dual}) \\
 &\quad + 2\bar{\eta}_j(\hat{R}'_{ab}(Q) + \sigma_{ab}\chi^i) - (\text{trace}), \\
 \delta \mathcal{R}_{ab}{}^{cd}(M) &= \bar{\varepsilon}_i \mathcal{D}\sigma_{cd}\hat{R}_{ab}{}^i{}_{\eta}(Q) - \frac{1}{2}\bar{\varepsilon}^i\sigma_{cd}\mathcal{R}_{abi}(S) \\
 &\quad - \bar{\eta}_i(2\sigma_{ab}\hat{R}'_{cd}(Q) + \sigma_{cd}\hat{R}'_{ab}(Q) + \sigma_{ab}\sigma_{cd}\chi^i) + \text{h.c.}, \\
 \delta(\mathcal{R}_{abi}(S) - \text{dual}) &= -2\mathcal{D}\varepsilon_j \hat{R}_{ab}{}^i{}_{\eta}(V) + \sigma_{cd}\mathcal{D}\mathcal{R}_{ab}{}^{cd}(M)\varepsilon_i \\
 &\quad - 4D_{[a}D_c T_{cb]ij}\varepsilon^j - \frac{1}{2}\bar{\chi}_j\gamma_c \hat{R}'_{ab}(Q)\gamma_c \varepsilon_i \\
 &\quad + 4(\frac{1}{2}\gamma_c T_{cdjk}D_d T_{ab}{}^{jk} + T_{ab}{}^{jk}\gamma_c D_d T_{cdjk} - T_{[ac}{}^{jk}\gamma_b]D_d T_{cdjk})\varepsilon_i \\
 &\quad - \sigma_{cd}\mathcal{R}_{ab}{}^{cd}(M)\eta_i - 2(\hat{R}_{ab}{}^i{}_{\eta}(V) - \hat{R}'_{ab}{}^i{}_{\eta}(V) \cdot \sigma\sigma_{ab})\eta_i \\
 &\quad + i\hat{R}(A) \cdot \sigma\sigma_{ab}\eta_i + 3\sigma_{ab}D\eta_i - (\text{dual}), \\
 \delta(D_{[a}D_c T_{cb]ij} - \text{dual}) &= \frac{1}{2}\bar{\varepsilon}_{[i}\mathcal{D}\mathcal{R}_{abj]}(S) - \bar{\varepsilon}_k\gamma_{[a}D_c(\chi^k T_{cb]ij}) \\
 &\quad - 2\bar{\varepsilon}_k\gamma_{[a}\hat{R}_{b]d}{}^k{}_{\eta}(Q)\tilde{D}_c T_{cdij} + \frac{1}{2}\bar{\eta}_{[i}\sigma_{ab}\sigma_{cd}\mathcal{R}_{cdj]}(S) - (\text{dual}).
 \end{aligned} \tag{5.8}$$

We see that only the Q -supersymmetry transformations of the high- θ components have developed non-linear terms [apart from (5.3), (5.4) and covariantizations]. This is very similar to the chiral scalar superfield [17], and is due to the fact that for the low- θ components no modifications can be found which have the correct Weyl and chiral weights. For the higher components such modifications are possible, and they appear for instance in the transformation rule for $\mathcal{R}(S)$. Note however that the S -supersymmetry transformations remain linear in the components of the Weyl multiplet, in accordance with the choice of the constraints.

Since certain transformation properties have nonlinear modifications, we expect such modifications in the Bianchi identities as well. The following complete set of Bianchi identities shows that this is indeed the case (because of different constraints, the full Bianchi identities do not coincide with those discussed for $N = 4$):

$$\varepsilon^{abcd} D_b \hat{R}_{cd}(D) = \varepsilon^{aec d} \mathcal{R}_{cd}{}^{be}(M), \quad (5.9)$$

$$\varepsilon^{abcd} D_b \hat{R}_{cd}(A) = 0, \quad (5.10)$$

$$\varepsilon^{abcd} D_b \hat{R}_{cd}{}^i{}_j(V) = 0, \quad (5.11)$$

$$\begin{aligned} \varepsilon^{abcd} D_b \mathcal{R}_{cd}{}^{ef}(M) = & \varepsilon^{abc[e} \hat{R}_{bc}{}^{f]}(K) + \frac{1}{4} \bar{\chi}_I \gamma_{[e} \chi^I \delta^a{}_{f]} + \frac{1}{2} (\bar{\chi}^I \gamma_a \hat{R}_{efI}(Q) \\ & + 2D_b(T_{abI} T_{ef}{}^I) - \text{h.c.}), \end{aligned} \quad (5.12)$$

$$\varepsilon^{abcd} D_b \hat{R}_{cd}{}^e{}_f(K) = \frac{1}{2} \bar{\chi}^I \gamma_a D_c \hat{R}_{ceI}(Q) - \frac{1}{12} \bar{\tilde{R}}_{ae}{}^i{}_j(Q) \mathcal{D} \chi^I - \text{h.c.}, \quad (5.13)$$

$$\varepsilon^{abcd} D_b \hat{R}_{cd}{}^i{}_j(Q) = +\frac{1}{2} \varepsilon^{abcd} \gamma_b \mathcal{R}_{cd}{}^i{}_j(S) + \frac{1}{2} \sigma_{ab} D_b \chi^I, \quad (5.14)$$

$$\begin{aligned} \varepsilon^{abcd} D_b \mathcal{R}_{cd}{}^i{}_j(S) = & \gamma_b \sigma \cdot T^I \tilde{\mathcal{R}}_{abI}(S) + 2D_c T_{cb}{}^I (\hat{R}(Q)_{abI} + \delta_{ab} \chi_I) \\ & + \{ \hat{R}_{bc}{}^i{}_j(V) - \tilde{\hat{R}}_{bc}{}^i{}_j(V) - \frac{1}{2} i \delta^i{}_j (\hat{R}_{bc}(A) - \tilde{\hat{R}}_{bc}(A)) \} \gamma_b \hat{R}_{ac}{}^j{}_i(Q). \end{aligned} \quad (5.15)$$

From eqs. (5.1) and (5.9)–(5.15) one can derive the complete non-linear equivalent of (5.4). These conditions can be compared with eqs. (2.6)–(2.18). We have (5.9) because of the constraint $\hat{R}_{\mu\nu}{}^a(P) = 0$. From (5.12) it is possible to obtain $\hat{R}_{ab}{}^c(K)$ in terms of $\mathcal{R}_{ab}{}^{cd}(M)$ [the non-linear analogue of (2.11)]. The Bianchi identity (5.13) then gives a condition with two covariant derivatives on $\mathcal{R}(M)$, which in the linear approximation reduces to (2.10). Eq. (5.15) has only non-linear terms on the right-hand side, and reduces to (2.18) on linearization.

Let us now consider the $N = 2$ superconformal action. We start with the observation that the square of the Weyl multiplet, contracted over all indices, is a chiral scalar multiplet with Weyl weight $w = 2$. Such chiral multiplets were discussed in detail in ref. [17]. They are of particular interest, since the highest component, a Lorentz scalar, has Weyl weight $w = 4$, and can therefore be used for the construction of an invariant action. The complete density formula was presented in [17]. We now apply this to the Weyl multiplet. Its square can be obtained from the product of two

superfields (5.2), i.e.

$$W^2 = \varepsilon_{ij}\varepsilon_{kl}W_{ab}{}^{ij}W_{ab}{}^{kl}, \quad (5.16)$$

and the correspondence (5.5). The highest component of the chiral superfield W^2 equals

$$C = 8\varepsilon^{ij}\varepsilon_{kl}A_{ab}{}^{kl}C_{abij} + 2B_{ab}{}^iB_{ab}{}^j + 8\bar{\psi}_{ab}{}^i\chi_{abi} + 4F_{ab}{}^{cd}F_{ab}{}^{cd}. \quad (5.17)$$

We can use (5.5), substitute the components of the Weyl multiplet into (5.16), (5.17), and apply the density formula of [17]. The complete $N = 2$ superconformal action is obtained from the real part of the resulting density. It is:

$$\begin{aligned} e^{-1}\mathcal{L} = & \frac{1}{2}(\hat{R}_{ab}(M) + 2T_{ac}{}^{ij}T_{bcij})(\hat{R}_{ba}(M) + 2T_{bd}{}^{kl}T_{adkl}) \\ & - 2\bar{\psi}_{[ak}\gamma_c\gamma_b\psi_{dl}T_{cd]}{}^{kl} + 8\bar{\psi}_{dk}\sigma_{db}\psi_{ei}T_{ae}{}^{kl}) \\ & - \frac{1}{6}\hat{R}(M)^2 - \frac{1}{2}\hat{R}(M)\bar{\psi}_{ci}\gamma_a\gamma_c\psi_{bj}T_{ab}{}^{ij} \\ & + 4D_aT_{ab}{}^{ij}(D_cT_{cbij} - \frac{1}{2}T_{cdij}\bar{\psi}_b{}^k\gamma_c\psi_{dk} - \frac{1}{2}T_{cbij}\bar{\psi}_d{}^k\gamma^d\psi_{ck}) \\ & - \frac{1}{2}(\bar{\chi}'\gamma_b - 2\bar{\psi}_j \cdot \gamma T_{ab}{}^{ij}\gamma_a)\mathcal{D}_b\chi_i - (1/e)\varepsilon^{\mu\nu\lambda\rho}\bar{\phi}_{\mu i}\gamma_\nu\mathcal{D}_\lambda\phi_\rho{}^i \\ & + \frac{3}{2}D^2 + \frac{1}{2}(\hat{R}(V))^2 - \frac{1}{8}(\hat{R}(A))^2 \\ & + 2(\hat{R}_{ab}{}^i(Q)\gamma_c + T_{ab}{}^{jk}\bar{\psi}_{ck})(\psi_{bi}\hat{R}_{ac}{}^j(V) - \frac{1}{2}\psi_{ci}\hat{R}_{ab}{}^j(V)) \\ & - \frac{1}{2}i\bar{\psi}_a{}^i\gamma_b\chi_i(\hat{R}_{ab}(A) - \frac{1}{2}\hat{R}_{ab}(A)) - \frac{1}{2}i\bar{\psi}_{[ci}\gamma_c\gamma_d\psi_{bj}T_{ab]}{}^{ij}\hat{R}_{ad}(A) \\ & + \frac{1}{2}\varepsilon^{abcd}\{\hat{R}_{ab}(M)(\frac{1}{2}\bar{\psi}_c{}^i\gamma_d\chi_i - \bar{\psi}_c{}^i\phi_{di}) + 2\bar{\psi}_a{}^i\phi_{bi}\hat{R}_{cd}{}^j(V) \\ & + \frac{1}{2}i\bar{\psi}_a{}^i\phi_{bi}\hat{R}_{cd}(A) + 4\bar{\phi}_{ai}\gamma_e\psi_{bj}D_cT_{ed}{}^{ij} \\ & + \frac{1}{2}(\bar{\psi}_a{}^i\sigma_{ef}\phi_{bi})(\bar{\psi}_c{}^j\sigma_{ef}\phi_{dj}) - \bar{\phi}_{ai}(\gamma_b\chi_j + 2\phi_{bj})T_{cd}{}^{ij} \\ & + \bar{\psi}_{ai}\phi_{bj}(\bar{\psi}_c{}^k\sigma \cdot T^{ij}\phi_{dk} - \bar{\chi}'\gamma_c\phi_d{}^i) \\ & + \bar{\psi}_a{}^i\gamma_e\psi_{bj}(2\bar{\phi}_{ci}\hat{R}_{ed}{}^j(Q) + \frac{1}{4}\bar{\phi}_{ci}\gamma_e\gamma_d\chi^j) \\ & - \bar{\psi}_a{}^i\gamma_e\psi_{bi}\bar{\phi}_{ej}\hat{R}_{cd}{}^j(Q) - 2\bar{\psi}_{[e}{}^i\gamma^e\psi_{a]}{}_i\bar{\phi}_{bj}\hat{R}_{cd}{}^j(Q)\} \\ & + \frac{1}{4}\bar{\psi}_a{}^i\sigma_{ab}\psi_b{}^j\bar{\chi}_i\chi_j + \bar{\psi}_a{}^i\psi_b{}^j(\frac{3}{8}\bar{\chi}_i\sigma_{ab}\chi_j + 2\bar{\psi}_{[a}{}^k\gamma_c]T_{bcij}\chi_k + \bar{\psi}^k \cdot \gamma T_{abij}\chi_k) \\ & + T_{abij}T_{bc}{}^{ij}(\bar{\phi}_{ak}\psi_c{}^k + \bar{\phi}_{ak}\sigma_{dc}\psi_d{}^k - 2\bar{\phi}_{dk}\sigma_{dc}\psi_a{}^k) \\ & - 4(\bar{\psi}_i \cdot \gamma T_{a\mu}{}^{ij} + \frac{1}{2}\bar{\psi}_{[ai}\gamma_cT_{\mu]}{}^{ij})\mathcal{D}_a\phi_{\mu j} + \text{h.c.} \end{aligned} \quad (5.18)$$

To put the action in this form, we have substituted for the dependent field $f_\mu{}^a$ the solution of the third constraint of (5.1):

$$f_\mu{}^a = \frac{1}{2}\hat{R}_{\mu a}(M) - \frac{1}{4}i\hat{R}_{\mu a}(A) + T_{\mu b}{}^{ij}T_{abij} - \frac{1}{4}e_\mu{}^a(D + \frac{1}{3}\hat{R}(M)). \quad (5.19)$$

Here $\hat{R}_{\mu b}(M)$ is equal to $\hat{R}_{\mu\nu}{}^{bc}(M)e^\nu{}_c$, with f put equal to zero, but with its

covariantizations. Furthermore we have performed a partial integration to obtain the Rarita–Schwinger term for the field $\phi_{\mu i}$. This term, as well as several others, can be compared directly with the $N = 1$ superconformal action given in [1]. It is argued in [1] that the superconformal action for $N = 1$ must be quadratic in the Poincaré field equations, and indeed the $N = 1$ action explicitly satisfies this requirement. This property presumably holds also for $N = 2$, although its presence is not manifest in (5.18). However, (5.18) does vanish on the use of the Poincaré field equations, and we expect the quadratic character of the action to become explicit after further algebraic manipulation.

Finally, one can compare our result with the linearized form of the $N = 2$ superconformal action [2]. As in [2], we expect a term of the form

$$-\frac{1}{3}\bar{\chi}^i\gamma^a\mathcal{D}_a\chi_i + \text{h.c.} \quad (5.20)$$

This term appears if we substitute for the dependent field $\phi_{\mu i}$ the expression

$$\phi_{\mu i} = \gamma^\lambda\mathcal{D}_{[\lambda}\psi_{\mu]i} + \frac{1}{2}\gamma^\lambda\sigma\cdot T_{ij}\gamma_\mu\psi_{\lambda}^j + \frac{1}{6}\gamma_\mu(\chi_i - \sigma\cdot T_{ij}\gamma\cdot\psi^j + 4\sigma^{\lambda\nu}\mathcal{D}_\lambda\psi_{\nu i}), \quad (5.21)$$

which is the solution of the second constraint of (5.1).

The results of this section can be extended to all $N \leq 4$. For this it would be useful to have a density formula, analogous to the one presented for $N = 2$ in [17], for $N = 3$ and $N = 4$. There are no essential difficulties in completing this program.

6. Conclusions

In this paper we have presented conformal supergravity theories for $N \leq 4$. The complete non-linear transformation rules and the corresponding commutator algebra of the superconformal gauge transformations were constructed for $N = 4$. This was done in a formulation with $SU(1, 1) \times U(1)$ invariance.

The next step in this program is the construction of the corresponding Poincaré supergravity theories. We expect that these theories can be derived by means of a suitable gauge choice from conformal multiplets. Such a procedure has been applied successfully to $N = 1$ and $N = 2$ supergravity [4, 5, 28]. It requires knowledge of a variety of superconformal multiplets, which can be found by implementation of the superconformal algebra on representations of rigid supersymmetry. At present not much is known about $N = 4$ multiplets (for an example, see [29]) and this subject deserves further study.

It is in principle straightforward to construct the complete actions of superconformal gravity for all $N \leq 4$. We have given the action for $N = 2$; its construction is greatly facilitated by the fact that a rather complete multiplet calculus is known in this case [4, 5, 16, 17]. There seems no essential difficulty to extend all known results of $N = 1$ and $N = 2$ conformal supergravity to the case of $N = 4$. It remains an intriguing question whether conformal supergravity will exist for $N > 4$ in spite of the apparent difficulties mentioned in the introduction.

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Appendix A

THE $N=4$ MULTIPLY OF CURRENTS

We consider the abelian version of the $N=4$ supersymmetric Yang–Mills theory [19], which contains a gauge field V_μ , a quartet of Majorana spinors ψ^i , taken with positive chirality, and a Lorentz scalar ϕ_{ij} , antisymmetric in $SU(4)$ indices and subject to an $SU(4)$ -covariant reality constraint:

$$\phi^{ij} \equiv (\phi_{ij})^* = \frac{1}{2} \varepsilon^{ijkl} \phi_{kl}. \quad (A.1)$$

The field strength of V_μ is denoted by $F_{\mu\nu}$, with self-dual components $F_{\mu\nu}^\pm \equiv \frac{1}{2}(F_{\mu\nu} \pm \tilde{F}_{\mu\nu})$. We will use a chiral notation throughout.

The fields transform under four independent supersymmetries with parameters ε^i :

$$\begin{aligned} \delta V_\mu &= \bar{\varepsilon}^i \gamma_\mu \psi_i + \text{h.c.}, \\ \delta F_{\mu\nu}^- &= \bar{\varepsilon}_i \tilde{\sigma}_{\mu\nu} \psi^i, \\ \delta \psi^i &= -\sigma \cdot F^- \varepsilon^i - 2i \tilde{\sigma} \phi^{ij} \varepsilon_j, \\ \delta \phi_{ij} &= i \bar{\varepsilon}_{[i} \psi_{j]} - i \varepsilon_{ijkl} \bar{\varepsilon}^k \psi^l. \end{aligned} \quad (A.2)$$

These transformations are an invariance of the action corresponding to the following lagrangian:

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{2} \bar{\psi}_i \tilde{\sigma} \psi^i - \frac{1}{2} (\partial_\mu \phi^{ij})(\partial_\mu \phi_{ij}). \quad (A.3)$$

Furthermore (A.3) is invariant under translations and chiral $SU(4)$ transformations. Note that the transformations (A.2) do not allow for an extra chiral $U(1)$ invariance.

The invariances of (A.3) lead to corresponding Noether currents: the energy momentum tensor $\theta_{\mu\nu}$, the supersymmetry currents $J_{\mu i}$ and the chiral $SU(4)$ currents v_{μ}^i . They are obtained in the standard way:

$$\begin{aligned} \theta_{\mu\nu} &= \frac{1}{2} \{ \delta_{\mu\nu} (F_{\rho\sigma}^-)^2 - 4 F_{\mu\rho}^- F_{\nu\sigma}^- + \text{h.c.} \} - \frac{1}{2} \bar{\psi}^i \gamma_{(\mu} \tilde{\sigma}_{\nu)} \psi_i + \delta_{\mu\nu} (\partial_\rho \phi^{ij})(\partial_\rho \phi_{ij}) \\ &\quad - 2 (\partial_\mu \phi^{ij})(\partial_\nu \phi_{ij}) - \frac{1}{3} (\square \delta_{\mu\nu} - \partial_\mu \partial_\nu) (\phi^{ij} \phi_{ij}), \\ J_{\mu i} &= -\sigma \cdot F^- \gamma_\mu \psi_i + 2i \phi_{ij} \tilde{\sigma}_\mu \psi^j + \frac{4}{3} i \sigma_{\mu\lambda} \partial_\lambda (\phi_{ij} \psi^j), \\ v_{\mu}^i &= \phi^{ik} \tilde{\sigma}_\mu \phi_{kj} + \bar{\psi}^i \gamma_\mu \psi_j - \frac{1}{4} \delta^i_j \bar{\psi}^k \gamma_\mu \psi_k. \end{aligned} \quad (A.4)$$

Note that v_{μ}^i is antihermitian and traceless.

The currents are conserved if the fields satisfy the equations of motion corresponding to the lagrangian (A.3). We have added some form-conserved improvement terms to $\theta_{\mu\nu}$ and $J_{\mu i}$ such that these currents satisfy further algebraic identities. Therefore we have the following equations:

$$\begin{aligned}\partial_\mu \theta_{\mu\nu} &= 0, & \partial_\mu J_{\mu i} &= 0, \\ \theta_{\mu\nu} &= \theta_{\nu\mu}, & \gamma_\mu J_{\mu i} &= 0, \\ \theta_{\mu\mu} &= 0, & \partial_\mu v_{\mu i} &= 0.\end{aligned}\tag{A.5}$$

When we now apply supersymmetry transformations on the currents (A.4), always using the field equations, we find a full supermultiplet of operators bilinear in the fields. In addition to the currents it contains the following quantities:

$$\begin{aligned}c &= (F_{ab}^-)^2, \\ \lambda_i &= \sigma \cdot F^- \psi_i, \\ e_{ij} &= \bar{\psi}_i \psi_j, \\ t_{ab}{}^{ij} &= \bar{\psi}^i \sigma_{ab} \psi^j + 2i \phi^{ij} F_{ab}^-, \\ \chi^i{}_k &= \frac{1}{2} \varepsilon^{ijmn} (\phi_{mn} \psi_k + \phi_{kn} \psi_m), \\ d^i{}_{kl} &= \phi^{ij} \phi_{kl} - \frac{1}{12} \delta^{[i}{}_k \delta^{j]}{}_l \phi^{mn} \phi_{mn}.\end{aligned}\tag{A.6}$$

Note that these quantities satisfy certain algebraic restrictions. The currents (A.4) and the operators (A.6) constitute the multiplet of currents [10, 11]. Taking into account the conditions (A.5) this supermultiplet contains $128+128$ degrees of freedom.

The supersymmetry transformations for the multiplet of currents are as follows:

$$\begin{aligned}\delta c &= 2\bar{\varepsilon}_i \delta \lambda^i, \\ \delta \lambda_i &= c^* \varepsilon_i + \delta e_{ik} \varepsilon^k - \sigma \cdot t_{ik} \tilde{\delta} \varepsilon^k, \\ \delta e_{ij} &= \bar{\varepsilon}_{(i} \lambda_{j)} + \frac{2}{3} i \varepsilon_{mnk(i} \bar{\varepsilon}^k \delta \chi_{j)}{}^{mn}, \\ \delta t_{ab}{}^{ij} &= -\frac{1}{2} \varepsilon^{ijkl} \bar{\varepsilon}_k \gamma^\rho \sigma_{ab} J_{\rho l} - \bar{\varepsilon}^{[i} \sigma_{ab} \lambda^{j]} + \frac{2}{3} i \varepsilon^{ijkl} \bar{\varepsilon}_n \delta \sigma_{ab} \chi^n{}_{kl}, \\ \delta \chi^i{}_k &= \frac{3}{4} i \varepsilon^{ijmn} \sigma \cdot t_{km} \varepsilon_n + \frac{3}{4} i \varepsilon^{ijmn} e_{mk} \varepsilon_n \\ &\quad - \frac{3}{4} i \gamma \cdot v^{[i}{}_k \varepsilon^{j]} + \frac{3}{2} i \delta d^i{}_{kl} \varepsilon^l - (\text{trace}), \\ \delta d^i{}_{kl} &= \frac{4}{3} i \bar{\varepsilon}_{[k} \chi^i{}_{l]} - \frac{2}{3} i \delta^{[i}{}_k \bar{\varepsilon}_n \chi^{nj]}{}_{l]} + \text{h.c.}, \\ \delta \theta_{\mu\nu} &= \bar{\varepsilon}^k \sigma_{(\mu\lambda} \partial_\lambda J_{\nu)k} + \text{h.c.}, \\ \delta J_{\mu i} &= -\gamma_\nu \theta_{\mu\nu} \varepsilon_i - 2(\gamma_\rho \sigma_{\mu\lambda} - \frac{1}{3} \sigma_{\mu\lambda} \gamma_\rho) \partial_\lambda v_{\rho i} \varepsilon^k \\ &\quad - (\sigma_{ab} \sigma_{\mu\lambda} + \frac{1}{3} \sigma_{\mu\lambda} \sigma_{ab}) \varepsilon_{iklm} \partial_\lambda t_{ab}{}^{kl} \varepsilon_m, \\ \delta v_{\mu i} &= -\bar{\varepsilon}^j J_{\mu j} + \frac{1}{4} \delta^j{}_i \bar{\varepsilon}^k J_{\mu k} + \frac{8}{3} i \bar{\varepsilon}^k \sigma_{\mu\lambda} \partial_\lambda \chi^i{}_{kj} - \text{h.c.}\end{aligned}\tag{A.7}$$

From the multiplet of currents one can straightforwardly construct a corresponding multiplet of fields. This is described in sect. 2, where we outline how the field representation for the $N = 4$ superconformal theory can be obtained in this way. The correspondence between most of these fields, listed in table 3, and the current multiplet components is rather obvious. We should add that the energy-momentum tensor couples to the metric tensor $g_{\mu\nu}$. The transformation properties of $g_{\mu\nu}$ then determine the transformations of the vierbein field modulo possible Lorentz transformations.

Appendix B

INVARIANCE GROUPS OF SUPERMULTIPLETS

The invariance groups of supermultiplets follow from the invariance of the supersymmetry algebra

$$\{Q'_\alpha, \bar{Q}'_\beta\} = -2iP_{\alpha\beta}\delta''.$$
 (B.1)

We disregard the possibility of central charges. The generators Q are Majorana spinors

$$\bar{Q}'_\alpha = C_{\alpha\beta}^{-1}Q'_\beta.$$
 (B.2)

Choosing γ -matrix conventions as in ref. [30] this implies

$$Q'_4 = (Q'_1)^\dagger, \quad Q'_2 = -(Q'_3)^\dagger.$$
 (B.3)

We now express the generators in terms of operators q_\pm and their hermitian conjugates, which are eigenstates of the chirality operator γ_5 :

$$\begin{aligned} q_+{}^i &= \frac{1}{2}(Q'_1 - Q'_3), \\ q_{-i} &= \frac{1}{2}(Q'_1 + Q'_3). \end{aligned}$$
 (B.4)

Notice that we have assigned upper and lower chiral $U(N)$ indices to the q 's according to the chiral notation used throughout this paper. The hermitian conjugates of the q 's are written as

$$\begin{aligned} q_{+i} &\equiv (q_+{}^i)^\dagger, \\ q_-{}^i &\equiv (q_{-i})^\dagger. \end{aligned}$$
 (B.5)

In terms of the q 's the supersymmetry algebra in a special choice of Lorentz frame takes the form

$$\begin{aligned} \text{massive case:} \quad p_\mu &= (0, 0, 0, iM), \\ \{q_+{}^i, q_{+j}\} &= \{q_-{}^i, q_{-j}\} = M\delta^i_j, \end{aligned}$$
 (B.6)

$$\begin{aligned} \text{massless case:} \quad p_\mu &= (0, 0, \omega, i\omega), \\ \{q_-{}^i, q_{-j}\} &= 2\omega\delta^i_j, \end{aligned}$$
 (4.7)

with all other anticommutators vanishing. Note the manifest invariance of (B.6), (B.7) under chiral $U(N)$.

The subsequent discussion deals with massive representations. In this case the little group is formed by rotations, which are generated by the σ -matrices on all pairs $p' = (q_+^i, -q_-^i)$ and $\tilde{p}_i = (q_{-i}, q_{+i})$. Notice that the two sets of pairs are related by hermitian conjugation, and that spin rotations commute with chiral $U(N)$ transformations.

We will now investigate whether there exists a larger group of linear transformations that leave the algebra (B.6) invariant, and that commute with spin rotations. Obviously the second requirement implies that such infinitesimal transformations can be described by a $(2N) \times (2N)$ matrix, which acts identically on the two $(2N)$ -dimensional vectors \mathbf{R}_1 and \mathbf{R}_2 formed from the first and second entries of the pairs p' and \tilde{p}_i :

$$\begin{aligned}\mathbf{R}_1 &= (q_+^1, q_+^2, \dots, q_+^N, q_{-1}, \dots, q_{-N}), \\ \mathbf{R}_2 &= (-q_-^1, -q_-^2, \dots, -q_-^N, q_{+1}, \dots, q_{+N}).\end{aligned}\quad (\text{B.8})$$

The relation between \mathbf{R}_1 and \mathbf{R}_2 is expressed by

$$\mathbf{R}_2 = -\Omega \mathbf{R}_1^\dagger \quad (\text{B.9})$$

where Ω is decomposed in $N \times N$ submatrices according to

$$\Omega = \begin{pmatrix} \theta & \mathbb{1} \\ -\mathbb{1} & \theta \end{pmatrix}. \quad (\text{B.10})$$

Since \mathbf{R}_1 and \mathbf{R}_2 should transform equally under the desired transformations, whose infinitesimal form is

$$\delta \mathbf{R} = \mathbf{X} \mathbf{R}, \quad (\mathbf{R} = \mathbf{R}_1 \text{ or } \mathbf{R}_2), \quad (\text{B.11})$$

the relation between \mathbf{R}_1 and \mathbf{R}_2 implies that the matrix \mathbf{X} satisfies the identity

$$\mathbf{X} \Omega = \Omega \mathbf{X}^*. \quad (\text{B.12})$$

In terms of \mathbf{R}_1 and \mathbf{R}_2 the supersymmetry algebra (B.6) has the form

$$\begin{aligned}\{\mathbf{R}_1, \mathbf{R}_1\} &= \{\mathbf{R}_2, \mathbf{R}_2\} = 0, \\ \{\mathbf{R}_1, \mathbf{R}_2\} &= \Omega.\end{aligned}\quad (\text{B.13})$$

Invariance of (B.13) under the transformations (B.11) requires

$$\mathbf{X} \Omega + \Omega \mathbf{X}^\top = 0. \quad (\text{B.14})$$

Combined with eq. (B.9) this implies that \mathbf{X} is antihermitian

$$\mathbf{X}^\dagger = -\mathbf{X}. \quad (\text{B.15})$$

Hence (B.14) and (B.15) show that the largest invariance group that leaves the supersymmetry algebra invariant and that commutes with spin rotations is the group

$\text{Sp}(2N)$. From our construction it is obvious that the chiral $U(N)$ group is a subgroup of $\text{Sp}(2N)$. The $U(N)$ decomposition of the $(2N)$ -dimensional vectors is

$$2N = N + \bar{N}. \quad (\text{B.16})$$

The invariance of the algebra will be valid for its representations. Therefore states of massive supermultiplets can be classified according to representations of $\text{Sp}(2N)$ for every given spin. This property, which has first been proven by Ferrara [24], is a useful tool in the construction of massive supermultiplets without central charges*.

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